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Modélisation et analyse mathématique d'écoulements complexes :
Modèles moyennés et milieux granulaires

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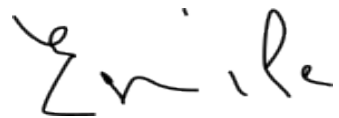
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Affidavit

Je soussigné, Emile Deléage, déclare par la présente que le travail présenté dans ce manuscrit est mon propre travail, réalisé sous la direction scientifique de Thierry Faug et Charlotte Perrin, dans le respect des principes d'honnêteté, d'intégrité et de responsabilité inhérents à la mission de recherche. Les travaux de recherche et la rédaction de ce manuscrit ont été réalisés dans le respect à la fois de la charte nationale de déontologie des métiers de la recherche et de la charte AMU relative à la lutte contre le plagiat.

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5. Journées scientifiques du RT Terre et Energie, Nouans-le-Fuzelier, 2024
6. Congrès des Jeunes Chercheur.e.s en Mathématiques Appliquées, Lyon, 2024 (exposé)
7. Sixth workshop on compressible multiphase flows, Strasbourg, 2024 (poster)
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Résumé et mots clés

Cette thèse est dédiée à la dérivation et à l'analyse mathématique de modèles d'écoulements complexes. La motivation principale de ce travail est la description des phénomènes d'avalanches. Ainsi, on s'intéresse à des modèles moyennés sur la profondeur, et/ou décrivant des écoulements granulaires. Dans une première partie, on étudie la structure des équations avec tenseur de Reynolds modélisant des fluides parfaits. On caractérise tout d'abord la symétrisabilité des équations moyennées de Reynolds. On étudie ensuite l'hyperbolicité de systèmes d'équations décrivant des écoulements diphasiques. On montre que la présence du tenseur de Reynolds permet de régulariser les équations. Dans une seconde partie, on présente un modèle à trois équations moyennées sur la profondeur décrivant un écoulement granulaire incompressible. En particulier, on obtient le premier modèle moyenné consistant à l'ordre 1 avec la rhéologie $\mu(I)$. On examine les prédictions du modèle en étudiant l'instabilité des ondes de surfaces. On donne ensuite une version régularisée de ce modèle, et on montre sa pertinence dans des régimes non stationnaires et non uniformes, grâce à des comparaisons avec des données expérimentales. On propose enfin un modèle consistant avec une généralisation compressible de la rhéologie $\mu(I)$, permettant de prendre en compte des effets de dilatance. Dans une troisième partie, on montre la stabilité asymptotique d'une onde progressive partiellement congestionnée pour un modèle jouet de suspension granulaire. Le caractère granulaire du milieu est ici modélisé via une viscosité effective qui diverge lorsque l'écoulement s'approche du régime congestionné.

Mots clés : modélisation, équations aux dérivées partielles, écoulements cisailés, écoulements peu profonds, milieux granulaires, écoulements multiphasiques, systèmes hyperboliques

Abstract and keywords

This thesis is dedicated to the derivation and analysis of mathematical models for complex flows. The main motivation of this work is the description of avalanches. As a consequence, the study focuses on depth-averaged models, and/or models of granular flows. In a first part, the structure of Reynolds equations for perfect fluids is studied. A characterization of the symetrisability of the equations is given. The hyperbolicity of systems of equations describing two-phase flows is then investigated. It is shown that the presence of the Reynolds tensor regularizes the equations. In a second part, a three-equation depth-averaged model for an incompressible granular flow is presented. In particular, the first depth-averaged model consistent up to order 1 with the $\mu(I)$ -rheology is obtained. The predictions of the models for the roll waves instability are examined. A regularized version of the model is then given, which is shown to be pertinent in non stationary and non uniform regimes through comparison with experimental data. Finally, a model consistently derived from a compressible generalization of the $\mu(I)$ -rheology and enabling to take dilatancy effects into account is presented. In a third part, a proof of the asymptotic stability of partially congested profiles for a granular suspension toy model is provided. In this model, granular effects are taken into account via a singular effective viscosity, that diverges when the flow reaches a congested regime.

Keywords: modelling, partial differential equations, shear flows, shallow water flows, granular media, multi-phase flows, hyperbolic systems

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Table des matières

Affidavit	2
Liste de publications et participation aux conférences	3
Résumé et mots clés	4
Abstract and keywords	5
Remerciements	6
Table des matières	8
Introduction	12
1. Équations générales de la mécanique des fluides	14
1.1. Trajectoires	15
1.2. Bilan de masse	18
1.3. Quantité de mouvement	20
1.4. Énergie	23
1.5. Équations moyennées sur la profondeur	27
2. Fluides parfaits	32
2.1. Pression	33
2.2. Outils pour l'analyse des écoulements compressibles	37
2.3. Tenseur de Reynolds	42
2.4. Principe de Hamilton	48
3. Rhéologies granulaires	56
3.1. Lois de friction	56
3.2. Rhéologie frictionnelle	62
3.3. Généralisations compressibles de la rhéologie $\mu(I)$	70
3.4. Rhéologie des suspensions	75
 I. Étude de la structure des équations avec tenseur de Reynolds modélisant des fluides parfaits	 80
1. Symétrisation des équations moyennées de Reynolds	81
1. Introduction and main results	81
2. Proof of the theorem	84

2. Modèles hyperboliques d'écoulements diphasiques turbulents	89
1. Introduction and main results	89
1.1. Different multiphase flow models and hyperbolicity criterion . .	90
1.2. Variational formulation of fluid mechanics	91
1.3. Models studied and main results	94
2. Derivation of the model from variational principle	101
2.1. Trajectories and Lagrangian quantities	102
2.2. Action and variations	103
2.3. Computation of the equation of the velocities	106
3. Hyperbolicity of the equations	108
3.1. Case $d = 1$	110
3.2. Case $d = 2$	112
3.3. Case $d = 3$	115
4. Other models	116
4.1. Case where the turbulence is carried by one phase	117
4.2. Case where each phase is turbulent	120
4.3. Other systems	123
5. Appendix	127
5.1. Computation of the characteristic polynomial	127

II. Modèles d'avalanche **134**

3. Modèle consistant d'écoulement granulaire incompressible	135
1. Introduction	136
2. Governing equations	139
2.1. Constitutive equations	139
2.2. Dimensionless form of the equations	142
3. Asymptotic expansion up to the first order	144
3.1. Order 0	145
3.2. Order 1	146
4. Derivation of the depth-averaged equations	148
4.1. Averaged variables	148
4.2. Depth-averaged mass and momentum equations	150
4.3. Depth-averaged energy equation	153
5. Properties and extensions of the model	155
5.1. Reconstruction of the velocity field	155
5.2. Hyperbolicity	156
5.3. Linear stability	157
5.4. Adding diffusive effects	161
5.5. Comparisons with the model of Gray and Edwards 2014	167
5.6. Numerical resolution	170
6. Limitations of the model	172
7. Conclusion	175

8.	Appendix	177
8.1.	Derivation of the depth-averaged energy equation	177
8.2.	Expression of the critical value of λ	179
8.3.	Critical Froude number for two-equation models	180
8.4.	Diffusive terms in the energy balance equation	181
4.	Régularisation et validation du modèle d'écoulement granulaire incompressible	183
1.	Introduction	183
2.	Presentation of the model	186
2.1.	A brief recall of the model of Deleage and Richard 2025	186
2.2.	Regularised model	189
3.	Study of dry fronts	194
4.	Avalanche stopping	199
5.	Accelerated flows	202
6.	Conclusion	204
5.	Modèle consistant d'écoulement granulaire compressible	205
1.	Introduction	205
2.	Governing equations	208
2.1.	Constitutive equations	208
2.2.	Dimensionless form of the equations	212
3.	Asymptotic expansion up to the first order	214
3.1.	Order 0	214
3.2.	Order 1	216
4.	Derivation of the depth-averaged equations	219
4.1.	Averaged variables	219
4.2.	Depth-averaged mass and momentum equations	222
4.3.	Depth-averaged energy equation	226
5.	Model analysis	228
5.1.	Hyperbolicity	228
5.2.	Formula for the height and reconstruction of the velocity field	230
5.3.	Linear stability and second order extension	232
5.4.	Domain of well-posedness	237
6.	Conclusion	237
7.	Appendix	238
7.1.	Case of a non-vanishing bulk friction coefficient	238
7.2.	Derivation of the depth-averaged energy equation	239

III. Étude mathématique d'un modèle jouet de suspension **242**

6. Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension	243
1. Introduction	243
1.1. Presentation of the model	243
1.2. Main results	246
2. Properties of the travelling wave solutions	250
2.1. Existence and asymptotic behavior of the travelling waves	250
2.2. Integrated system and reformulation of the equations	252
2.3. Preliminary estimates	253
3. Well-posedness for the integrated system	255
3.1. Basic energy estimate	255
3.2. Formulation of the system for higher order estimates	257
3.3. Construction of global strong solutions	258
3.4. Stability of the ball B_δ	259
3.5. The map $\mathcal{A}^\epsilon$ is a contraction	270
4. Asymptotic stability of (u_ϵ, v_ϵ)	276
4.1. Stability of the profile $u - u_\epsilon$	276
4.2. Bounding $(v - v_\epsilon)$ in $L^1(\mathbb{R})$	281
4.3. Concluding the proofs of Theorem 6.2 and Corollary 6.1	284
5. Appendix	285
5.1. Proof that the map $\mathcal{A}^\epsilon$ is well defined	285
5.2. Proof of Lemma 6.5	287
Conclusion	292
Bibliographie	296

Introduction

Les avalanches de neige, de blocs rocheux et de glace constituent un aléa important en montagne, qui risque d'augmenter avec le réchauffement climatique (dégel du permafrost, instabilités et ruptures glaciaires). Afin d'évaluer les risques et de mettre en œuvre des mesures de protection adéquates, il est important de disposer de modèles mathématiques fiables permettant des simulations numériques d'événements dans des temps de calcul raisonnables. Cette thèse est dédiée à la dérivation et à l'analyse de modèles mathématiques adaptés à la description des avalanches de neige sèche. Les outils utilisés pour cela sont ceux de la mécanique des fluides, qui décrit l'écoulement de milieux considérés comme continus. Plus précisément, on suppose souvent que les avalanches se comportent comme un fluide complexe, par opposition à des fluides comme l'eau liquide ou l'air, dont le comportement est en général plus simple. Pour écrire des modèles précis et pertinents, il est nécessaire de prendre en compte les spécificités de la situation étudiée. Dans cette thèse, les efforts de modélisation sont principalement portés sur les deux axes suivants :

Modèles moyennés

Afin d'obtenir un modèle peu coûteux en temps de calcul et donc facilement utilisable en pratique, une attention particulière est portée sur les modèles d'équations moyennées sur la profondeur. Ces modèles permettent en effet de réduire de un la dimension d'espace, et donc d'être rapidement résolus numériquement. L'utilisation de tels modèles pour la description des avalanches peut être justifiée par les propriétés physiques des avalanches. En effet, la longueur typique d'une avalanche est généralement très supérieure à sa hauteur typique. Le prix à payer de cette approche est une éventuelle perte d'information (par exemple, la vitesse exacte est remplacée par la vitesse moyenne), qui peut se traduire à terme par une perte de précision. Pour contourner ce problème, on présente dans cette thèse des modèles augmentés, avec une variable supplémentaire, obtenue de manière consistante avec le processus de moyennisation et permettant de stocker des informations additionnelles. Plus précisément, on considère des modèles avec tenseur de Reynolds, ce qui permet dans ce contexte de connaître le cisaillement vertical de l'écoulement. La structure mathématique de modèles de mécanique des fluides avec tenseur de Reynolds est étudiée dans la première partie de la thèse, et fait l'objet de deux résultats. Le premier résultat concerne la symétrisabilité des équations en dimensions 2 et 3. Le second résultat concerne la dérivation de modèles multiphasiques avec tenseur de Reynolds.

Milieus granulaires

Un autre aspect dont la prise en compte est cruciale pour la modélisation est la nature du matériau considéré et sa manière de s'écouler. En ce qui concerne les avalanches de neige, on distingue deux grandes familles : les avalanches de neige sèche et les avalanches de neige humide. On se concentre ici sur la neige sèche, qui peut être décrite physiquement par un matériau granulaire, c'est-à-dire par un milieu constitué d'un grand nombre de particules solides. Dans ces avalanches, la neige peut se diviser en une couche dense, située près du sol, et une couche diluée formant un aérosol dans la partie supérieure de l'avalanche. Plus généralement, les milieux granulaires peuvent exhiber des comportements très variés, assimilables selon les régimes à un matériau solide, liquide ou gazeux. On se focalise ici sur la description de la partie plus dense de l'avalanche, pour laquelle on néglige le rôle de l'air. Les modèles obtenus dans ce contexte peuvent donc a priori être appliqués à d'autres types d'écoulements géophysiques (éboulements rocheux, sable). Dans ce régime dense, des effets collectifs complexes sont causés par les nombreuses interactions mécaniques entre les particules (collisions, friction). On se restreint ici à la description de ces phénomènes purement mécaniques, et on ne prend pas en compte les effets thermiques. Un phénomène physique déterminant pour la description des milieux granulaires dense est la friction entre les particules. Dans un modèle de milieu continu, la friction peut être modélisée par une rhéologie adaptée, qui décrit la déformation du milieu sous l'effet de contraintes. Diverses rhéologies frictionnelles ont été proposées empiriquement pour les milieux granulaires. Mathématiquement, l'analyse des équations correspondantes peut se révéler complexe, et ces rhéologies sont parfois mal posées. La deuxième partie de cette thèse est consacrée à la dérivation de modèles moyennés sur la profondeur pour un écoulement granulaire sec. Les modèles obtenus fournissent une approximation à la fois consistante et bien posée d'une rhéologie granulaire (rhéologie $\mu(I)$). Des validations des modèles par comparaison avec des mesures expérimentales sont effectuées. Dans la troisième partie, on étudie le caractère bien-posé d'un modèle jouet inspiré d'une rhéologie de suspension granulaire, la rhéologie $\mu(J)$, qui décrit un milieu granulaire immergé dans une phase liquide en considérant à la fois la friction entre les particules et des interactions hydrodynamiques. Les équations considérées ont également été obtenues par homogénéisation et modélisent des effets de congestion, aussi appelée jamming, qui apparaissent lorsque la concentration des particules solides dans le milieu approche un seuil critique. Le résultat clef présenté dans cette partie est la stabilité asymptotique, dans un espace d'énergie adapté, d'une solution de type onde progressive modélisant un front partiellement congestionné.

Le reste de cette introduction est dédié à une présentation détaillée des différents outils utilisés dans cette thèse, ce qui permet d'énoncer les résultats obtenus de manière plus précise tout en les situant dans un contexte plus global. En section 1, on rappelle les équations de la mécanique des fluides et des écoulements moyennés sur la profondeur. La section 2 se concentre sur l'étude structurelle des équations des

fluides parfaits, i.e. pour lesquels les effets de friction ne sont pas pris en compte, et le rôle joué par le tenseur de Reynolds. Deux résultats de la thèse sont présentés dans les paragraphes 2.3 et 2.4. Enfin, la section 3 est dédiée à la présentation de rhéologies frictionnelles utilisées dans la modélisation des écoulements granulaires. Des résultats de la thèse sont présentés dans les paragraphes 3.2, 3.3 et 3.4.

1. Équations générales de la mécanique des fluides

Dans cette section, on donne une présentation générale des équations de la mécanique des fluides, et on montre comment appliquer ces équations dans le cas des écoulements moyennés sur la profondeur. Cette section a uniquement pour but d'introduire les différents outils mathématiques qui sont courants en mécanique des fluides et qui seront utilisés tout au long de la thèse. En particulier, aucun résultat original n'est énoncé dans cette section. Les personnes familières avec le formalisme de la mécanique des fluides peuvent donc directement passer à la section 2.

Le point de vue adopté ici pour présenter ce formalisme est celui des équations bilans. Les bilans jouent en effet un rôle important en modélisation, car ils permettent de relier entre elles deux échelles cruciales, appelées ici échelles macroscopique et mésoscopique (LEFEVRE et al. 2022) :

1. L'échelle macroscopique correspond aux grandeurs physiques qui peuvent être mesurées sur le terrain ou en laboratoire. Ces mesures concernent donc des objets de taille finie, c'est-à-dire de longueur/surface/volume non nuls. Dans le contexte des milieux granulaires, la précision des mesures effectuées en laboratoire est typiquement de l'ordre du dixième de millimètre.
2. L'échelle mésoscopique est une échelle à laquelle le système étudié peut être considéré comme un milieu continu. Les grandeurs mésoscopiques sont définies localement, c'est-à-dire en tout point de l'espace et du temps. Cette échelle est donc particulièrement adaptée pour écrire des équations d'évolution, puisque l'état du milieu est connu avec une infinie précision. On l'appelle ici mésoscopique car elle se situe entre les échelles macroscopique et microscopique (qui considère chaque particule individuellement). Un volume mésoscopique doit donc être suffisamment important pour contenir un grand nombre de particules, et suffisamment petit pour être négligeable devant les volumes macroscopiques considérés.

On modélise donc le système grâce à des équations locales décrivant l'évolution de grandeurs mésoscopiques. Ces équations sont dérivées à l'aide de bilans, écrits à l'échelle macroscopique à partir des lois de la physique. Réciproquement, ces bilans permettent ensuite d'obtenir l'évolution des grandeurs physiques macroscopiques à partir des équations locales. Si les équations locales sont exactes et fermées, elles demandent en revanche de connaître la solution en tout point, ce qui peut en pratique nécessiter beaucoup de temps de calcul et n'est pas forcément utile selon les applications. Au contraire, si on se contente d'écrire des équations pour les grandeurs

macroscopiques qu'on veut modéliser, on n'obtient pas en général un système d'équations fermé, et on doit donc procéder à des approximations pouvant induire une perte de précision. Dans le contexte des écoulements en couche mince, un bon compromis est de considérer une échelle intermédiaire, qui est l'échelle des écoulements moyennés sur la profondeur. Les équations correspondantes peuvent être obtenues à partir des équations locales en faisant des bilans partiels et sont introduites à la fin de cette section. On commence par présenter la notion de trajectoires au paragraphe 1.1, ce qui permet d'utiliser les points de vue eulérien et lagrangien. Les paragraphes 1.2, 1.3 et 1.4 sont respectivement consacrés aux équations de la masse, de la quantité de mouvement et de l'énergie. Le paragraphe 1.5 est dédié aux équations moyennées sur la profondeur.

1.1. Trajectoires

1.1.1. Points de vue eulérien et lagrangien

À l'échelle mésoscopique, un fluide est un milieu continu, occupant un domaine $\Omega \in \mathbb{R}^d$. Son mouvement est donc décrit par un continuum de trajectoires, notées $x(t, X) \in \Omega$. Le label $X \in \Omega$ est la position initiale de la trajectoire à un instant donné (ici $t = 0$) et permet de distinguer les différentes trajectoires entre elles. Il peut être interprété comme une étiquette qui suit chaque "particule fluide" et est appelé coordonnée lagrangienne. Les quantités qui dépendent de la coordonnée X sont appelées quantités lagrangiennes.

Exemple 1. *La vitesse lagrangienne est définie par*

$$v^l(t, X) := \partial_t x(t, X). \quad (0.1)$$

C'est la vitesse de la particule fluide qui était à la position X à l'instant $t = 0$. De même, on peut définir l'accélération lagrangienne

$$a^l(t, X) := \partial_t v^l(t, X) = \partial_t^2 x(t, X). \quad (0.2)$$

Ce sont les mêmes définitions qu'en mécanique du point.

Cependant, en mécanique des fluides, il est souvent plus utile de connaître les propriétés du fluide à un endroit de l'espace qui reste fixe à mesure que le temps passe, plutôt que de suivre individuellement chaque trajectoire. C'est le point de vue eulérien. À chaque variable lagrangienne $f^l(t, X)$, on peut associer la variable eulérienne correspondante notée $f(t, x)$ par

$$f^l(t, X) = f(t, x(t, X)). \quad (0.3)$$

On supposera que pour tout instant t , l'application $X \mapsto x(t, X)$ est un difféomorphisme, d'inverse noté $x \mapsto X(t, x)$. La donnée $X(t, x)$ est la position initiale d'une particule fluide se trouvant en position x à l'instant t . Elle permet d'exprimer un

champ eulérien en fonction du champ lagrangien correspondant :

$$f(t, x) = f^l(t, X(t, x)). \quad (0.4)$$

Exemple 2. La vitesse eulérienne est définie par

$$v(t, x) = v^l(t, X(t, x)), \quad \text{ou de manière équivalente,} \quad v^l(t, X) = v(t, x(t, X)). \quad (0.5)$$

En dérivant cette dernière expression par rapport au temps, on obtient

$$a(t, x(t, X)) = a^l(t, X) = \partial_t v^l(t, X) = [\partial_t v + (v \cdot \nabla) v](t, x(t, X)), \quad (0.6)$$

d'où l'expression de l'accélération eulérienne :

$$a(t, x) = [\partial_t v + (v \cdot \nabla) v](t, x). \quad (0.7)$$

Il est important d'avoir à l'esprit le fait que les quantités f et f^l représentent la même grandeur physique. La différence est que la variable f^l indique la valeur de cette grandeur pour chaque particule, alors que la variable f indique cette grandeur à chaque position $x \in \Omega$. Il est donc plus adéquat de considérer le passage de f à f^l comme un changement de variables plutôt que comme un changement de fonction.

D'une manière générale, il est plus simple d'exprimer les lois de la physique dans les coordonnées lagrangiennes. En effet, le point de vue lagrangien permet de considérer un système fermé (i.e. qui n'échange pas de matière avec l'extérieur). Au contraire, en eulérien, les particules considérées en un point ne sont jamais les mêmes. Dans la suite, on va donc écrire d'abord les équations du mouvement dans les variables lagrangiennes, pour ensuite les transposer dans les coordonnées eulériennes.

1.1.2. Transport

Supposons qu'une quantité f soit transportée par le fluide. Autrement dit, f est constante le long des trajectoires. En coordonnées lagrangiennes, cela s'écrit simplement

$$f^l(t, X) = f^l(0, X). \quad (0.8)$$

En dérivant cette équation par rapport au temps, on obtient

$$0 = \partial_t f^l(t, X) = [\partial_t f + v \cdot \nabla f](t, x(t, X)). \quad (0.9)$$

On appelle équation de transport l'équation

$$\partial_t f + v \cdot \nabla f = 0. \quad (0.10)$$

Formellement, l'équation de transport (0.10) est l'équivalent eulérien de (0.8). Comme on se place généralement du point de vue eulérien, c'est la forme (0.10) qui est la plus utilisée. Notons cependant que l'équation (0.8) a du sens même si f n'est pas dérivable. Par extension, on peut dire que f est solution d'une équation de transport même

quand f n'est pas une fonction régulière. Notons cependant que le passage entre les points de vue eulérien et lagrangien n'est pas toujours évident pour des fonctions non régulières (voir par exemple WAGNER 1987 pour un exemple de justification de ce passage dans un cas particulier).

Exemple 3. On dit qu'un ensemble $\omega(t) \subset \Omega$ est transporté par le fluide s'il existe $\omega_0 \subset \Omega$ tel que $\omega(t) = x(t, \omega_0)$. Mathématiquement, cela revient à dire que la fonction indicatrice

$$\mathbf{1}_\omega(t, x) := \begin{cases} 1, & \text{si } x \in \omega(t), \\ 0, & \text{sinon,} \end{cases} \quad (0.11)$$

est solution de l'équation de transport (0.10). En effet, on a $\mathbf{1}_\omega^l = \mathbf{1}_{\omega_0}$. En physique, on dit que le fluide contenu dans $\omega(t)$ est un système fermé.

Lorsqu'une quantité f varie le long de la trajectoire, on connaît souvent son taux d'évolution grâce à la physique du modèle. On peut alors écrire une version plus générale de l'équation de transport (0.10), dans laquelle le zéro du membre de droite est remplacé par un terme source S , qui représente ici le taux d'évolution de f le long de la trajectoire :

$$\partial_t f + v \cdot \nabla f = S. \quad (0.12)$$

Exemple 4. Considérons un écoulement à surface libre tel que représenté sur la Figure 1. On note $x \in \mathbb{R}^{d-1}$ et $z \in \mathbb{R}$ (resp. $u \in \mathbb{R}^{d-1}$ et $w \in \mathbb{R}$) les coordonnées (resp. vitesses) horizontale et verticale. On note $h(t, x) \in \mathbb{R}$ la hauteur de fluide à l'instant t au point x . L'évolution de h est donnée par la condition cinématique suivante :

$$\partial_t h + u(h) \cdot \nabla h = w(h), \quad \text{avec } f(h)(t, x) := f(t, x, h(t, x)). \quad (0.13)$$

Cette condition cinématique sert de condition aux limites dans les problèmes d'écoulement à surface libre. On voit que la hauteur h est solution d'une équation de transport avec terme source. Le membre de gauche signifie que la hauteur est transportée à la vitesse horizontale évaluée à la surface libre $u(h)$. Le membre de droite signifie que la dérivée de la hauteur le long de la trajectoire est donnée par la vitesse verticale du fluide évaluée à la hauteur h .

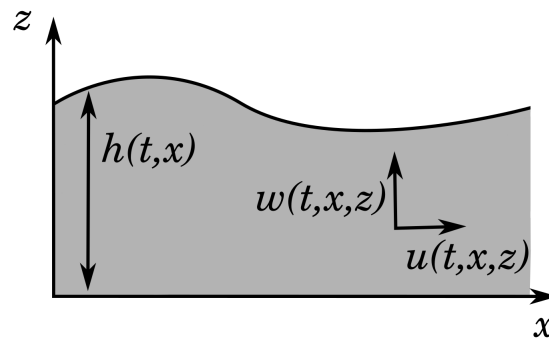


FIGURE 1. – Schéma d'écoulement à surface libre

1.2. Bilan de masse

1.2.1. Masse et densité

La masse d'un volume de fluide est une quantité qui peut être définie comme la résistance à la modification de sa vitesse par une force extérieure. En effet, on peut réécrire le principe fondamental de la dynamique pour une particule ponctuelle de masse m comme

$$a = \frac{F}{m}. \quad (0.14)$$

Ainsi, si on applique une force F donnée à une particule de masse m , on voit que plus la masse de la particule est importante, plus son accélération est faible.

Pour un milieu continu, la masse dépend du volume considéré. On note $M(t, \omega)$ la masse du fluide contenu dans ω à l'instant t . La masse vérifie les propriétés suivantes :

- $M(t, \omega) \geq 0$,
- $M(t, \emptyset) = 0$,
- Si les ensembles $\{\omega_n\}_{n \in \mathbb{N}}$ sont deux à deux disjoints, alors

$$M\left(t, \bigsqcup_{n \in \mathbb{N}} \omega_n\right) = \sum_{n \in \mathbb{N}} M(t, \omega_n).$$

On dit que la masse est une grandeur extensive.

Mathématiquement, ces trois propriétés signifient que, pour tout t , l'application $\omega \mapsto M(t, \omega)$ est une mesure. Cette mesure satisfait une propriété supplémentaire, qui vient du fait que le fluide est décrit comme un milieu continu :

- Si $|\omega| = 0$, alors $M(t, \omega) = 0$, où $|\omega|$ désigne le volume de ω .

On dit que $M(t, \cdot)$ est absolument continue par rapport à la mesure de Lebesgue. Le théorème de Radon-Nikodym implique alors qu'il existe une fonction $\rho(t, x) \geq 0$ telle que, pour tous t, ω ,

$$M(t, \omega) = \int_{\omega} \rho(t, x) dx. \quad (0.15)$$

Le champ ρ est appelé la densité du fluide (en théorie de la mesure, cela correspond à la dérivée de Radon-Nikodym de la masse par rapport à la mesure de Lebesgue). On peut également définir la densité lagrangienne ρ^l par

$$\rho^l(t, X) := \rho(t, x(t, X)). \quad (0.16)$$

1.2.2. Conservation

Le principe de conservation de la masse énonce que la masse de fluide contenue dans un ensemble $\omega(t)$ transporté par le fluide est constante :

$$M(t, \omega(t)) = M(0, \omega_0), \quad \text{soit} \quad \int_{\omega(t)} \rho(t, x) dx = \int_{\omega_0} \rho(0, x) dx. \quad (0.17)$$

En faisant le changement de variables $x = x(t, X)$, avec $x(0, X) = X$, on obtient

$$\int_{\omega_0} \rho^l(t, X) J(t, X) dX = \int_{\omega_0} \rho^l(0, X) dX, \quad \text{où } J := \det \frac{\partial x}{\partial X} > 0 \quad (0.18)$$

(la positivité de J est assurée par le fait que $J(0, X) = 1$ et que J ne s'annule pas). Comme ω_0 est arbitraire, on en déduit la version lagrangienne de l'équation locale de conservation de la masse :

$$\rho^l J = \rho|_{t=0}. \quad (0.19)$$

En dérivant par rapport au temps et en utilisant la formule $\partial_t J = J(\operatorname{div} v)^l$, on en déduit l'équation de conservation de la masse dans les coordonnées eulériennes, ou simplement équation de conservation :

$$\partial_t \rho + \operatorname{div}(\rho v) = 0. \quad (0.20)$$

Plus généralement, en reprenant ce calcul, on peut démontrer la proposition suivante :

Proposition 1. (Théorème de transport de Reynolds) Si $\omega(t) \in \Omega$ est transporté par le fluide, alors pour tout champ $\varphi = \varphi(t, x) \in \mathbb{R}$, on a

$$\frac{d}{dt} \int_{\omega(t)} \varphi(t, x) dx = \int_{\omega(t)} [\partial_t \varphi + \operatorname{div}(\varphi v)](t, x) dx. \quad (0.21)$$

Remarque 1. Formellement, on peut retrouver la formule ci-dessus en utilisant l'équation de transport satisfaite par $\mathbf{1}_{\omega}$. En effet, on a pour tout t

$$\begin{aligned} \frac{d}{dt} \int_{\omega(t)} \varphi dx &= \frac{d}{dt} \int_{\mathbb{R}^d} \varphi \mathbf{1}_{\omega(t)} dx = \int_{\mathbb{R}^d} (\partial_t \varphi \mathbf{1}_{\omega(t)} + \varphi \partial_t \mathbf{1}_{\omega(t)}) dx \\ &= \int_{\mathbb{R}^d} (\partial_t \varphi \mathbf{1}_{\omega(t)} - \varphi v \cdot \nabla \mathbf{1}_{\omega(t)}) dx = \int_{\mathbb{R}^d} \mathbf{1}_{\omega(t)} [\partial_t \varphi + \operatorname{div}(\varphi v)] dx \\ &= \int_{\omega(t)} [\partial_t \varphi + \operatorname{div}(\varphi v)] dx. \end{aligned}$$

Exemple 5. En prenant $\varphi(t, x) = 1$, on obtient que la dérivée du volume d'un ensemble $\omega(t) \subset \Omega$ transporté par le fluide est donnée par

$$\frac{d}{dt} |\omega(t)| = \frac{d}{dt} \int_{\omega(t)} dx = \int_{\omega(t)} \operatorname{div}(v) dx \quad (0.22)$$

Cela donne une interprétation supplémentaire de l'équation de conservation de la masse. En effet, on peut la réécrire comme une équation de transport avec terme source :

$$\partial_t \rho + v \cdot \nabla \rho = -\rho \operatorname{div}(v). \quad (0.23)$$

Si par exemple $\operatorname{div}(v) > 0$, la formule (0.22) implique que le volume occupé par le fluide augmente. L'équation (0.23) indique que la densité doit diminuer le long des trajectoires. La compensation entre l'augmentation du volume et la diminution de la densité permet

à la masse de rester constante.

Définition 1. On appelle écoulement incompressible un écoulement pour lequel on a en tout point $\operatorname{div}(v) = 0$. Dans le cas contraire, on parle d'écoulement compressible.

Remarque 2. Par l'exemple précédent, considérer un écoulement incompressible revient à dire que le volume de tout ensemble transporté par le fluide est constant. Dans ce cas, (0.23) implique que la densité ρ est transportée par le fluide. Une solution particulière de (0.23) pour un écoulement incompressible est une densité ρ constante. Réciproquement, un écoulement pour lequel ρ est constante est nécessairement incompressible. On parle alors d'écoulement incompressible homogène. Si la densité ρ n'est pas constante, on parle d'écoulement incompressible inhomogène.

Remarque 3. Le principe de conservation de la masse a été énoncé d'un point de vue lagrangien (la masse d'un ensemble transporté par le fluide est constante). Grâce à l'équation de conservation (0.20), on peut exprimer ce principe d'un point de vue eulérien. En effet, si $\omega \subset \Omega$ est un domaine fixe, on a

$$\frac{d}{dt}M(t, \omega) = \frac{d}{dt} \int_{\omega} \rho dx = \int_{\omega} \partial_t \rho dx = - \int_{\omega} \operatorname{div}(\rho v) dx = - \int_{\partial\omega} \rho v \cdot n dS, \quad (0.24)$$

où la dernière égalité est obtenue en appliquant la formule de Stokes, avec $\partial\omega$ le bord de ω , n la normale sortante et dS la mesure de Hausdorff sur $\partial\omega$. On dit que la dérivée de la masse de fluide contenue dans ω est donnée par le flux de masse qui passe à travers $\partial\omega$. En effet, si on se place en un point $x \in \partial\omega$, la quantité $v \cdot n dS dt$ représente le volume infinitésimal dV de fluide qui sort de ω par le point x pendant dt (voir Figure 2). En multipliant par ρ , on obtient donc la masse infinitésimale qui sort à cet endroit pendant dt .

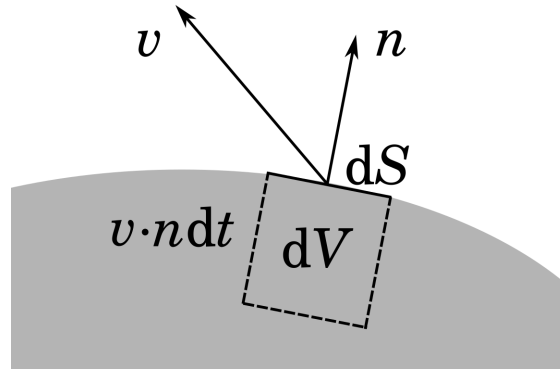


FIGURE 2. – Volume infinitésimal de fluide sortant de ω (en gris) pendant dt .

1.3. Quantité de mouvement

1.3.1. Bilan de quantité de mouvement

Dans la partie précédente, on a utilisé le bilan d'une quantité macroscopique (la masse) afin d'obtenir une équation d'évolution locale pour la densité (équation de

conservation). On voudrait appliquer cette même méthode afin d'obtenir une équation d'évolution pour la vitesse du fluide. Cependant, contrairement à la masse, la vitesse n'est pas une grandeur extensive. C'est pourquoi on va plutôt utiliser la quantité de mouvement, définie pour un ensemble $\omega \subset \Omega$ à un instant t par

$$Q(t, \omega) = \int_{\omega} \rho v dx. \quad (0.25)$$

Dans un volume $\omega(t) \subset \Omega$ transporté par le fluide, le principe fondamental de la dynamique s'écrit

$$\frac{d}{dt} Q(t, \omega(t)) = F_R, \quad (0.26)$$

avec F_R la résultante des forces s'appliquant au fluide contenu dans $\omega(t)$.

Remarque 4. En introduisant $\bar{v}$ la vitesse définie par

$$\bar{v}(t, \omega) = \frac{1}{M(t, \omega)} \int_{\omega} \rho v dx, \quad (0.27)$$

et en utilisant la conservation de la masse, le principe fondamental de la dynamique (0.26) peut se réécrire

$$M(t, \omega(t)) \frac{d}{dt} \bar{v}(t, \omega(t)) = F_R. \quad (0.28)$$

On retrouve une formulation analogue à celle de la mécanique du point. La vitesse $\bar{v}$, qui peut être vue comme une vitesse moyenne, est plus précisément la vitesse du centre de gravité x_G de l'ensemble transporté par le fluide, défini par

$$x_G(t, \omega) = \frac{1}{M(t, \omega)} \int_{\omega} \rho x dx. \quad (0.29)$$

En effet, le théorème de transport de Reynolds couplé avec la conservation de la masse donne

$$\begin{aligned} \frac{d}{dt} x_G(t, \omega(t)) &= \frac{1}{M(t, \omega(t))} \int_{\omega(t)} [\partial_t(\rho x) + \operatorname{div}(\rho v \otimes x)^T] dx \\ &= \frac{1}{M(t, \omega(t))} \int_{\omega(t)} [(\partial_t \rho + \operatorname{div}(\rho v))x + (\rho v \cdot \nabla)x] dx = \bar{v}(t, \omega(t)). \end{aligned}$$

Pour exprimer la résultante des forces F_R , on distingue deux types de forces : les forces volumiques, qui s'exercent à l'intérieur du volume $\omega(t)$ et qui ont une cause extérieure au fluide, et les forces surfaciques, qui s'exercent sur le bord $\partial\omega(t)$ et qui sont causées par les interactions du fluide à l'extérieur de $\omega(t)$ sur le fluide qui se trouve dans $\omega(t)$.

Exemple 6. Le poids, noté F_P , est une force volumique qui est proportionnelle à la masse du fluide. Il s'écrit

$$F_P(t, \omega) = M(t, \omega)g = \int_{\omega} \rho(t, x)g dx, \quad (0.30)$$

avec $g \in \mathbb{R}^d$ le vecteur constant d'accélération de la pesanteur. C'est la seule force volumique considérée dans cette thèse.

Les forces surfaciques, notées F_S , peuvent s'écrire comme une intégrale sur le bord $\partial\omega(t)$:

$$F_S = \int_{\partial\omega(t)} s(t, x, n) dS, \quad (0.31)$$

avec $s(t, x, n) = s(n) \in \mathbb{R}^d$ la force par unité de surface s'exerçant sur le bord au point x et à l'instant t . En écrivant la résultante des forces comme $F_R = F_P + F_S$ et en appliquant la Proposition 1 à chaque coordonnée du vecteur de quantité de mouvement, on peut réécrire le principe fondamental de la dynamique comme

$$\int_{\omega(t)} [\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v)^T] dx = \int_{\omega(t)} \rho g dx + \int_{\partial\omega(t)} s(n) dS. \quad (0.32)$$

1.3.2. Tenseur des contraintes

Afin de déduire du principe fondamental de la dynamique une équation d'évolution locale, il faut exprimer la dernière intégrale du membre de droite de (0.32) comme une intégrale sur $\omega(t)$. C'est rendu possible par la proposition suivante :

Proposition 2. (*Existence du tenseur des contraintes*) Il existe un tenseur symétrique $\sigma = \sigma(t, x)$ tel que, pour tout vecteur n , on a $s(n) = \sigma n$.

La preuve de cette proposition peut s'effectuer en trois étapes (voir par exemple le livre de SARAMITO 2016).

- La première étape consiste à montrer la troisième loi de Newton, aussi appelé principe des actions réciproques, qui s'écrit simplement ici $s(-n) = -s(n)$. C'est une conséquence du principe fondamental de la dynamique et du fait que la quantité de mouvement est une grandeur extensive.
- La deuxième étape utilise la relation $s(-n) = -s(n)$ pour obtenir l'existence d'un tenseur σ tel que $s(n) = \sigma n$.
- La troisième étape est la preuve que $\sigma = \sigma^T$. C'est une conséquence du théorème du moment cinétique, qui stipule que, pour tout ensemble $\omega(t) \subset \Omega$ transporté par le fluide, on a

$$\frac{d}{dt} L(t, \omega(t)) = \int_{\omega(t)} x \wedge \rho g dx + \int_{\partial\omega(t)} x \wedge s(n) dS, \quad \text{où} \quad L(t, \omega) = \int_{\omega} x \wedge \rho v dx. \quad (0.33)$$

La quantité $L(t, \omega)$ est le moment cinétique du fluide contenu dans ω à l'instant t . C'est l'analogue de la quantité de mouvement pour les rotations.

Par la proposition 2 et la formule de Stokes, on peut réécrire la contribution des forces surfaciques :

$$\int_{\partial\omega(t)} s(n) dS = \int_{\partial\omega(t)} \sigma n dS = \int_{\omega(t)} \operatorname{div}(\sigma)^T dx. \quad (0.34)$$

Le principe fondamental de la dynamique (0.32) devient alors :

$$\int_{\omega(t)} [\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v)^T - \rho g - \operatorname{div}(\sigma)^T] dx = 0. \quad (0.35)$$

On en déduit l'équation locale de conservation de la quantité du mouvement :

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v)^T = \rho g + \operatorname{div}(\sigma)^T. \quad (0.36)$$

En utilisant l'équation de conservation de la masse (0.20), et en divisant l'équation (0.36) par ρ , on obtient l'équation d'évolution pour la vitesse :

$$\partial_t v + (v \cdot \nabla) v = g + \frac{1}{\rho} \operatorname{div}(\sigma)^T. \quad (0.37)$$

Remarque 5. On reconnaît dans le membre de gauche de (0.36) la structure de l'équation de conservation (0.20). On dit que la densité de quantité de mouvement ρv est une variable conservative. Les variables conservatives (comme ρ ou ρv) peuvent être associée à des grandeurs macroscopiques extensives (comme la masse M ou la quantité de mouvement $Q = M \bar{v}$). En revanche, l'équation (0.37) a une structure d'équation de transport. On dit que la vitesse est non conservative. Plus généralement, si une grandeur f satisfait une équation de transport

$$\partial_t f + v \cdot \nabla f = S, \quad (0.38)$$

alors le produit ρf satisfait l'équation de conservation

$$\partial_t(\rho f) + \operatorname{div}(\rho f v) = \rho S. \quad (0.39)$$

1.4. Énergie

L'énergie d'un système est une grandeur fondamentale. En physique, elle permet de caractériser l'état du système. D'un point de vue mathématique, c'est une grandeur qui agit comme une norme pour les variables qui décrivent le fluide. Contrôler l'énergie est souvent crucial pour montrer que les équations sont bien posées, c'est-à-dire qu'il y a existence et unicité des solutions, ainsi qu'une dépendance continue de ces solutions par rapport à leur valeur initiale.

1.4.1. Énergie cinétique

L'énergie cinétique est la part d'énergie du fluide qui est liée à son mouvement. Pour obtenir une équation d'évolution locale de l'énergie cinétique, on peut partir de l'équation de conservation de la quantité de mouvement (0.36) et prendre le produit scalaire par la vitesse v . En utilisant l'équation de conservation de la masse, on arrive

à l'équation suivante :

$$\partial_t \left(\frac{1}{2} \rho |v|^2 \right) + \operatorname{div} \left(\frac{1}{2} \rho |v|^2 v \right) = \rho g \cdot v + \operatorname{div}(\sigma) v. \quad (0.40)$$

On reconnaît la structure de l'équation de conservation dans le membre de gauche. La quantité $\rho e_c := \rho |v|^2 / 2$ est appelée la densité d'énergie cinétique. Cela signifie qu'on peut définir l'énergie cinétique du fluide contenu dans un ensemble $\omega \subset \Omega$ au temps t par

$$\mathcal{E}_c(t, \omega) = \int_{\omega} \rho e_c(t, x) dx = \int_{\omega} \frac{1}{2} \rho |v|^2(t, x) dx. \quad (0.41)$$

En intégrant l'équation (0.40) sur un ensemble $\omega(t) \subset \Omega$ transporté par le fluide, on obtient avec le théorème de transport 1 la formule suivante :

$$\frac{d}{dt} \mathcal{E}_c(t, \omega(t)) = \int_{\omega(t)} \left[\partial_t \left(\frac{1}{2} \rho |v|^2 \right) + \operatorname{div} \left(\frac{1}{2} \rho |v|^2 v \right) \right] dx = \int_{\omega(t)} \rho g \cdot v dx + \int_{\omega(t)} \operatorname{div}(\sigma) v dx. \quad (0.42)$$

En utilisant la formule de Stokes et la symétrie du tenseur des contraintes, la dernière intégrale peut se réécrire comme

$$\int_{\omega(t)} \operatorname{div}(\sigma) v dx = \int_{\omega(t)} [\operatorname{div}(\sigma v) - \sigma : D(v)] dx = \int_{\partial\omega(t)} v \cdot \sigma n dS - \int_{\omega(t)} \sigma : D(v) dx, \quad (0.43)$$

avec

$$D(v) := \frac{1}{2} \left[\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x} \right)^T \right], \quad (0.44)$$

et $\partial v / \partial x$ la matrice jacobienne de la vitesse v . On en déduit le théorème de l'énergie cinétique pour un fluide :

$$\frac{d}{dt} \mathcal{E}_c(t, \omega(t)) = \int_{\omega(t)} \rho g \cdot v dx + \int_{\partial\omega(t)} v \cdot \sigma n dS - \int_{\omega(t)} \sigma : D(v) dx. \quad (0.45)$$

On voit que la dérivée de l'énergie cinétique du fluide contenu dans $\omega(t)$ s'exprime comme une somme de trois termes :

- Le premier terme correspond à la puissance associée au poids,
- Le second terme est la puissance des forces exercées par le fluide situé à l'extérieur de $\omega(t)$ sur le bord $\partial\omega(t)$. Il est parfois noté $\dot{W}$:

$$\dot{W} = \int_{\partial\omega(t)} v \cdot \sigma n dS. \quad (0.46)$$

- Enfin, le troisième terme est la puissance des forces que le fluide à l'intérieur de $\omega(t)$ exerce sur lui-même. Contrairement aux deux précédents, il ne correspond à aucune force intervenant dans le bilan de quantité de mouvement (0.32). En effet, par le principe des actions réciproques, la résultante des actions du fluide sur lui-même est nécessairement nulle. Cependant, la puissance associée à

ces interactions n'est pas égale à zéro en général. Cette puissance est appelée une puissance intérieure. Notons qu'ici, cette puissance vaut zéro si $D(v) = 0$. Comme expliqué par le lemme suivant, cela signifie qu'elle provient de la déformation du fluide.

Lemme 1. *Un milieu continu est en mouvement rigide, i.e. le champ de vitesse peut s'écrire comme la somme d'une translation et d'une rotation,*

$$v(t, x) = v_0(t) + \omega(t) \wedge x, \quad (0.47)$$

si, et seulement si, on a $D(v) = 0$.

Une preuve de ce lemme peut être trouvée par exemple dans le livre de Franck BOYER et al. 2012.

Remarque 6. *En utilisant la vitesse $\bar{v}$ du centre de gravité (0.27), on peut écrire la densité d'énergie cinétique comme*

$$\frac{1}{2}\rho|v|^2 = \frac{1}{2}\rho|\bar{v}|^2 + \rho(v - \bar{v}) \cdot \bar{v} + \frac{1}{2}\rho|v - \bar{v}|^2, \quad (0.48)$$

ce qui donne après intégration et en utilisant la définition de $\bar{v}$,

$$\mathcal{E}_c = \frac{1}{2}M|\bar{v}|^2 + \frac{1}{2}\int_{\omega} \rho|v - \bar{v}|^2 dx \quad (0.49)$$

L'énergie cinétique se décompose donc comme la somme de deux termes positifs. Le terme $M|\bar{v}|^2/2$ correspond à l'énergie cinétique de translation du fluide à vitesse $\bar{v}$. Le second terme correspond à l'énergie cinétique calculée dans le référentiel du centre de gravité et fait intervenir le second moment de la vitesse. En utilisant la conservation de la masse et le principe fondamental de la dynamique (0.28), on peut calculer la dérivée temporelle de l'énergie cinétique de translation à vitesse $\bar{v}$ dans un ensemble transporté par le fluide :

$$\frac{d}{dt} \frac{1}{2}M|\bar{v}|^2 = M \frac{d\bar{v}}{dt} \cdot \bar{v} = M\mathbf{g} \cdot \bar{v} + \int_{\partial\omega} \sigma n \cdot \bar{v} dS \quad (0.50)$$

En soustrayant cette équation au théorème de l'énergie cinétique (0.45), on en déduit que l'évolution du second moment de la vitesse est donnée par

$$\frac{d}{dt} \frac{1}{2}\int_{\omega(t)} \rho|v - \bar{v}|^2 dx = \int_{\partial\omega(t)} (v - \bar{v}) \cdot \sigma dS - \int_{\omega(t)} \sigma : D(v) dx. \quad (0.51)$$

Par conséquent, on voit qu'à l'échelle macroscopique, le bilan de quantité de mouvement et d'énergie cinétique sont indépendants :

- Le bilan de quantité de mouvement donne l'évolution de $\bar{v}$ (moment d'ordre 1 de la vitesse). Il permet donc de décrire l'écoulement moyen du fluide.
- Le bilan d'énergie cinétique donne l'évolution du moment d'ordre 2 de la vitesse. Il fait intervenir les fluctuations de vitesse, et joue donc un rôle crucial dans l'étude de la turbulence.

1.4.2. Énergie totale

En général, l'énergie cinétique n'est pas le seul type d'énergie présent au sein d'un fluide. En effet, il existe également d'autres types d'énergie : l'énergie potentielle et l'énergie interne. La prise en compte de ces grandeurs permet d'obtenir une loi de conservation supplémentaire. En effet, si on définit l'énergie totale comme la somme des énergies cinétique, potentielle et interne, on peut énoncer le premier principe de la thermodynamique, qui stipule que l'énergie totale d'un système isolé (i.e. qui n'échange ni matière, ni travail, ni chaleur avec l'extérieur) est constante. En d'autres mots, les puissances intervenant dans le théorème de l'énergie cinétique (0.45) peuvent se réécrire comme des dérivées temporelles de grandeurs qui sont interprétées comme des formes d'énergie additionnelles.

Exemple 7. En remarquant que l'accélération de la pesanteur g peut s'écrire comme $g = -\nabla(gz)$, avec g la norme du vecteur d'accélération de la pesanteur, tel que $g = -ge_z$, on obtient

$$v \cdot g = -v \cdot \nabla(gz) = -\partial_t(gz) - v \cdot \nabla(gz). \quad (0.52)$$

On reconnaît une structure d'équation de transport. En utilisant la remarque 5, on en déduit en multipliant par ρ que

$$\rho g \cdot v = -\partial_t(\rho gz) - \operatorname{div}(\rho gz v). \quad (0.53)$$

On dit que le poids est une force conservative. Le bilan local d'énergie (0.40) peut donc se réécrire

$$\partial_t \left(\frac{1}{2} \rho |v|^2 + \rho gz \right) + \operatorname{div} \left[\left(\frac{1}{2} \rho |v|^2 + \rho gz \right) v \right] = \operatorname{div}(\sigma) v. \quad (0.54)$$

ρgz est la densité d'énergie potentielle du fluide. On peut également définir l'énergie mécanique comme la somme de l'énergie cinétique et de l'énergie potentielle :

$$\mathcal{E}_m(t, \omega) = \int_{\omega} \left(\frac{1}{2} \rho |v|^2 + \rho gz \right) dx = \mathcal{E}_c(t, \omega) + \mathcal{E}_p(t, \omega). \quad (0.55)$$

En appliquant le théorème de transport 1, on obtient que la dérivée de l'énergie mécanique dans un système fermé $\omega(t) \subset \Omega$ est donnée par

$$\frac{d}{dt} \mathcal{E}_m(t, \omega(t)) = \int_{\partial\omega(t)} v \cdot \sigma n dS - \int_{\omega(t)} \sigma : D(v) dx. \quad (0.56)$$

L'énergie totale $\mathcal{E}_{tot}$ est définie comme la somme des énergies mécanique $\mathcal{E}_m$ et interne $\mathcal{E}_{int}$. Le premier principe de la thermodynamique énonce que dans un système fermé, la dérivée de l'énergie totale est donnée par la puissance des forces extérieures s'appliquant sur la paroi, plus le flux thermique entrant. Dans un ensemble $\omega(t) \in \Omega$ transporté par le fluide, cela se traduit par

$$\frac{d}{dt} \mathcal{E}_m(t, \omega(t)) + \frac{d}{dt} \mathcal{E}_{int}(t, \omega(t)) = \frac{d}{dt} \mathcal{E}_{tot}(t, \omega(t)) = \dot{W} + \dot{Q}, \quad (0.57)$$

où $\dot{W}$ est défini par l'équation (0.46) et $\dot{Q}$ est le flux thermique. Dans cette thèse, on se concentrera uniquement sur l'étude de modèles qui ne prennent pas en compte les effets thermiques, comme indiqué en début d'introduction. On supposera donc par la suite que $\dot{Q} = 0$. On en déduit que la dérivée de l'énergie interne est donnée par

$$\frac{d}{dt} \mathcal{E}_{int}(t, \omega(t)) = \int_{\omega(t)} \sigma : D(v). \quad (0.58)$$

Remarque 7. En introduisant ρe la densité d'énergie interne, telle que

$$\mathcal{E}_{int}(t, \omega) = \int_{\omega} \rho(t, x) e(t, x) dx, \quad (0.59)$$

on peut déduire de (0.58) l'équation locale d'évolution de l'énergie interne :

$$\partial_t(\rho e) + \text{div}(\rho e v) = \sigma : D(v). \quad (0.60)$$

On voit que le terme $\sigma : D(v)$ correspond à un transfert d'énergie mécanique en énergie interne. Ce terme peut être interprété comme un transfert d'échelle pour les fluctuations de vitesse. En effet, il apparaît avec un signe $-$ dans l'équation (0.51) des fluctuations de la vitesse mésoscopique, et avec un signe $+$ dans l'équation de l'énergie interne, qui est en général une fonction croissante de la température (pouvant être définie comme la moyenne des fluctuations de la vitesse microscopique).

1.5. Équations moyennées sur la profondeur

On s'intéresse dans cette section à un cas particulier, celui des équations moyennées sur la profondeur. Ces équations sont utilisées notamment pour modéliser des écoulements en couche mince, i.e. pour lesquels le ratio entre la hauteur typique de l'écoulement et sa longueur typique est petit. C'est souvent le cas des avalanches de neige, dont l'épaisseur est de l'ordre du mètre et la longueur peut atteindre plusieurs centaines de mètres.

1.5.1. Hypothèses de l'écoulement et adimensionnement

On considère un fluide bidimensionnel ($d = 2$) incompressible, de densité ρ constante, s'écoulant sur un plan faisant un angle θ avec l'horizontale et soumis à la gravité, comme représenté sur la Figure 3. On note $x \in \mathbb{R}$ et $z \in \mathbb{R}$ (resp. $u \in \mathbb{R}$ et $w \in \mathbb{R}$) les coordonnées (resp. vitesses) parallèle et perpendiculaire au plan incliné. On a donc $v = (u, w)^T$. La hauteur du fluide à l'instant t et à l'abscisse x est notée $h(t, x)$. Dans ces coordonnées, le tenseur des contraintes σ est noté

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{xz} & \sigma_{zz} \end{pmatrix}. \quad (0.61)$$

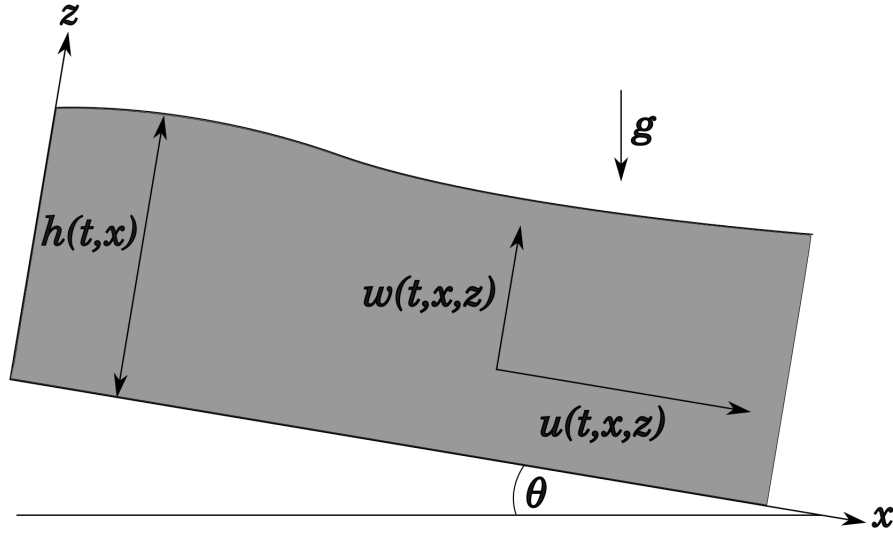


FIGURE 3. – Schéma d'écoulement sur plan incliné

Les équations de la masse et de la quantité de mouvement s'écrivent alors

$$\begin{cases} \partial_x u + \partial_z w = 0, \\ \partial_t u + \partial_x(u^2) + \partial_z(uw) = g \sin \theta + \frac{1}{\rho} \partial_x \sigma_{xx} + \frac{1}{\rho} \partial_z \sigma_{xz}, \\ \partial_t w + \partial_x(uw) + \partial_z(w^2) = -g \cos \theta + \frac{1}{\rho} \partial_x \sigma_{xz} + \frac{1}{\rho} \partial_z \sigma_{zz}. \end{cases} \quad (0.62)$$

Le bilan local d'énergie cinétique (0.40) devient

$$\partial_t e_c + \operatorname{div}(e_c v) = u \left(g \sin \theta + \frac{\partial_x \sigma_{xx}}{\rho} + \frac{\partial_z \sigma_{xz}}{\rho} \right) + w \left(\frac{\partial_x \sigma_{xz}}{\rho} + \frac{\partial_z \sigma_{zz}}{\rho} - g \cos \theta \right), \quad (0.63)$$

avec

$$\rho e_c = \frac{1}{2} \rho (u^2 + w^2) \quad (0.64)$$

la densité d'énergie cinétique. Ces équations sont valides à l'intérieur de l'écoulement, i.e. dans la zone définie par $0 < z < h(x, t)$ (zone en gris dans la Figure 3). Elle sont complétées par des conditions aux limites :

— en $z = 0$, conditions de non-glissement et non-pénétration :

$$u(0) = 0, \quad w(0) = 0. \quad (0.65)$$

— en $z = h$, la hauteur évolue selon la condition cinématique

$$\partial_t h + u(h) \partial_x h = w(h), \quad (0.66)$$

et la force extérieure exercée sur la surface libre est considérée nulle :

$$\sigma(h)n = 0, \quad \text{avec} \quad n = \frac{1}{\sqrt{1 + (\partial_x h)^2}} \begin{pmatrix} -\partial_x h \\ 1 \end{pmatrix} \quad (0.67)$$

le vecteur unitaire sortant normal à la surface libre.

Remarque 8. Ces conditions aux limites sont les conditions les plus simples et les plus fréquentes dans la littérature. On peut néanmoins trouver des conditions différentes suivant les contextes. Par exemple, la condition de non-glissement peut être remplacée par une condition de Navier (NAVIER 1823) lorsque la vitesse de glissement n'est pas nulle, typiquement sur un fond lisse. De même, la force extérieure exercée à la surface libre peut être modifiée en prenant en compte des effets tels qu'une pression extérieure, l'influence du vent ou la tension de surface (ZADRZYŃSKA 2004). Ces effets ne seront pas considérés ici.

Finalement, on fait l'hypothèse que l'écoulement est en couche mince. Pour mettre en évidence cette hypothèse, on va adimensionner les équations. On introduit donc h_0 la hauteur typique, L_0 la longueur typique de variation, et u_0 la vitesse horizontale typique. On peut alors introduire deux paramètres sans dimension : ε le paramètre de couche mince, et F le nombre de Froude. Ils sont définis par

$$\varepsilon := \frac{h_0}{L_0} \quad \text{et} \quad F := \frac{u_0}{\sqrt{gh_0}}. \quad (0.68)$$

L'hypothèse de couche mince (aussi appelée hypothèse des ondes longues) s'écrit alors $\varepsilon \ll 1$. Le nombre de Froude F exprime le rapport entre l'inertie de l'écoulement et la gravité. On supposera également que $F = O(1)$, ce qui correspond aux régimes d'avalanche étudiés expérimentalement et observés sur le terrain (F est typiquement entre 5 et 10, voir ISSLER 2003). On définit les variables adimensionnées par

$$\tilde{x} = \frac{x}{L_0}, \quad \tilde{z} = \frac{z}{h_0}, \quad \tilde{h} = \frac{h}{h_0}, \quad \tilde{u} = \frac{u}{u_0}, \quad \tilde{w} = \frac{w}{\varepsilon u_0}, \quad \tilde{t} = t \frac{u_0}{L_0}, \quad \tilde{\sigma} = \frac{\sigma}{\rho g h_0}. \quad (0.69)$$

Les équations (0.62) peuvent alors se réécrire

$$\begin{cases} \partial_{\tilde{x}} \tilde{u} + \partial_{\tilde{z}} \tilde{w} = 0, \\ \varepsilon F^2 [\partial_{\tilde{t}} \tilde{u} + \partial_{\tilde{x}} (\tilde{u}^2) + \partial_{\tilde{z}} (\tilde{u} \tilde{w})] = \sin \theta + \varepsilon \partial_{\tilde{x}} \tilde{\sigma}_{xx} + \partial_{\tilde{z}} \tilde{\sigma}_{xz}, \\ \varepsilon^2 F^2 [\partial_{\tilde{t}} \tilde{w} + \partial_{\tilde{x}} (\tilde{u} \tilde{w}) + \partial_{\tilde{z}} (\tilde{w}^2)] = -\cos \theta + \varepsilon \partial_{\tilde{x}} \tilde{\sigma}_{xz} + \partial_{\tilde{z}} \tilde{\sigma}_{zz}. \end{cases} \quad (0.70)$$

La densité d'énergie cinétique adimensionnée vaut

$$\tilde{e}_c = \frac{e_c}{u_0^2} = \frac{1}{2} (\tilde{u}^2 + \varepsilon^2 \tilde{w}^2). \quad (0.71)$$

Remarque 9. Dans ces coordonnées, on voit que

— L'équation de la masse reste inchangée,

- La composante horizontale du poids est compensée à l'ordre zéro par la divergence du tenseur des contraintes, les termes d'accélération horizontale apparaissent seulement comme une correction d'ordre un en ε .
- La composante verticale du poids est compensée à l'ordre 0 et 1 par la divergence du tenseur des contraintes, les termes d'accélération verticale apparaissent comme une correction d'ordre deux en ε .
- La contribution de la vitesse verticale à l'énergie cinétique est d'ordre ε^2 par rapport à la contribution de la vitesse horizontale.

Enfin, les conditions aux limites (0.65) et (0.66) restent inchangées dans les nouvelles variables, et la condition (0.67) devient

$$\begin{cases} \tilde{\sigma}_{xz}(\tilde{h}) - \varepsilon \tilde{\sigma}_{xx}(\tilde{h}) = 0, \\ \tilde{\sigma}_{zz}(\tilde{h}) - \varepsilon \tilde{\sigma}_{xz}(\tilde{h}) = 0. \end{cases} \quad (0.72)$$

1.5.2. Moyennisation sur la profondeur

On peut désormais moyenner les équations sur la profondeur, afin d'obtenir un nouveau système pour les grandeurs qui nous intéressent. Ces grandeurs dépendent uniquement des variables (t, x) puisque la variable z disparaît dans le processus de moyennisation. Les grandeurs les plus utilisées dans les modèles d'écoulement à couche mince sont la hauteur h et la vitesse horizontale moyenne U définie par

$$U(t, x) := \frac{1}{h(t, x)} \int_0^{h(t, x)} u(t, x, z) dz, \quad \tilde{U} := \frac{U}{u_0}. \quad (0.73)$$

Comme on l'a vu précédemment, la forme conservative des équations est la plus adaptée pour écrire des bilans macroscopiques, grâce au théorème de transport 1. C'est la même chose pour les équations moyennées sur la profondeur, qui peuvent s'interpréter comme un "bilan partiel", effectué sur une portion de fluide située au temps t à l'abscisse x , de largeur infinitésimale dx et de hauteur $h(t, x)$. Le lemme suivant est l'analogue du théorème de transport de Reynolds dans le contexte des écoulements moyennés sur la profondeur, et permet d'intégrer facilement des équations de conservation.

Lemme 2. Pour tout champ $f = f(t, x, z) \in \mathbb{R}$, l'intégrale de l'opérateur de conservation appliqué à f est donnée par

$$\int_0^h [\partial_t f + \operatorname{div}(f v)] dz = \partial_t \left(\int_0^h f dz \right) + \partial_x \left(\int_0^h f u dz \right). \quad (0.74)$$

En effet, on a

$$\begin{aligned} & \int_0^h [\partial_t f + \operatorname{div}(f v)] dz \\ &= \int_0^h [\partial_t f + \partial_x(f u) + \partial_z(f w)] dz \\ &= \partial_t \left(\int_0^h f dz \right) + \partial_x \left(\int_0^h f u dz \right) - f(h) [\partial_t h + u(h) \partial_x h - w(h)] - f(0) w(0) \end{aligned}$$

et la formule s'obtient en utilisant les conditions cinématique (0.66) et de non pénétration (0.65).

En utilisant le lemme 2 et l'équation de conservation de la masse, on obtient une équation d'évolution sur h , donnée par

$$\partial_t h + \partial_x(hU) = 0. \quad (0.75)$$

On reconnaît une équation de conservation, analogue à celle de la densité (0.20). En effet, comme indiqué par l'équation (0.22), l'incompressibilité implique que le volume du fluide ne change pas, ce qui se traduit ici par une conservation de la hauteur.

De même, on peut utiliser le lemme 2 pour moyenner l'équation de quantité de mouvement horizontale et obtenir

$$\partial_t(hU) + \partial_x \left(\int_0^h u^2 dz \right) = gh \sin \theta + \frac{1}{\rho} \left[\partial_x \left(\int_0^h \sigma_{xx} dz \right) - \sigma_{xx}(h) \partial_x h + \sigma_{xz}(h) - \sigma_{xz}(0) \right], \quad (0.76)$$

qu'on peut simplifier en utilisant la condition aux limites sur σ à la surface libre (0.67) :

$$\partial_t(hU) + \partial_x \left(\int_0^h u^2 dz \right) = gh \sin \theta + \frac{1}{\rho} \left[\partial_x \left(\int_0^h \sigma_{xx} dz \right) - \sigma_{xz}(0) \right]. \quad (0.77)$$

Pour calculer l'intégrale du carré de la vitesse horizontale, on peut procéder comme dans la décomposition de l'énergie cinétique (0.49) et utiliser la vitesse moyenne pour écrire :

$$\int_0^h u^2 dz = hU^2 + \int_0^h (u - U)^2 dz. \quad (0.78)$$

On obtient donc le système suivant de deux équations pour les variables h et U :

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x \left(hU^2 + \int_0^h (u - U)^2 dz \right) = gh \sin \theta + \frac{1}{\rho} \left[\partial_x \left(\int_0^h \sigma_{xx} dz \right) - \sigma_{xz}(0) \right]. \end{cases} \quad (0.79)$$

Ces équations sont exactes, mais elle ne sont pas fermées. Pour obtenir un système d'équations fermé, il faut pouvoir traiter les termes suivants :

1. Dans le membre de droite, les termes faisant intervenir le tenseur σ . La définition

du tenseur σ dépend du type de fluide étudié et fait l'objet des deux prochaines sections :

- Dans la section 2, on considère un fluide parfait, pour lequel le tenseur σ est donné par un unique scalaire, la pression ;
 - Dans la section 3, on considère des écoulements granulaires : granulaire sec et suspension. Le tenseur σ est alors plus complexe et est défini par une rhéologie frictionnelle.
2. Dans le membre de gauche, le second moment de la vitesse. Plusieurs méthodes sont possibles pour traiter ce terme :
- La méthode la plus répandue consiste à négliger ce terme (voir par exemple BRESCH 2009). C'est la méthode qui est utilisée pour obtenir les équations de Saint Venant (cf paragraphe 2.1.2).
 - La méthode utilisée dans cette thèse est de considérer ce terme comme une troisième variable, indépendante de h et de U , appelée enstrophie ou tenseur de Reynolds. L'équation de l'énergie est alors utilisée afin d'obtenir une troisième équation d'évolution pour cette variable. Cette méthode est décrite plus en détail dans la section 2.3 sur le tenseur de Reynolds.

Remarque 10. *Les calculs précédents peuvent aussi être effectués dans le cas d'un écoulement tridimensionnel. Aux coordonnées x et z s'ajoute alors la coordonnée transverse y . Le champ de vitesse est noté $v = (u_x, u_y, w)^T$. Par souci de concision, on note $v_h = (u_x, u_y)^T$ la vitesse horizontale et div_h l'opérateur divergence dans les coordonnées horizontales, tel que $\text{div}_h[(a_x, a_y)^T] = \partial_x a_x + \partial_y a_y$ (avec l'extension habituelle pour un tenseur d'ordre 2). Dans ce cas, les équations moyennés obtenues sont*

$$\begin{cases} \partial_t h + \text{div}_h(h V_h) = 0, \\ \partial_t(h V_h) + \text{div}_h \left(h V_h \otimes V_h + \int_0^h (v_h - V_h) \otimes (v_h - V_h) dz \right)^T \\ = \begin{pmatrix} gh \sin \theta \\ 0 \end{pmatrix} + \text{div}_h \left[\frac{1}{\rho} \int_0^h \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix} dz \right]^T - \frac{1}{\rho} \begin{pmatrix} \sigma_{xz}(0) \\ \sigma_{yz}(0) \end{pmatrix}. \end{cases} \quad (0.80)$$

2. Fluides parfaits

Un fluide parfait est un fluide qui s'écoule sans frottement interne, comme décrit plus en détail dans le paragraphe suivant. Dans le cas d'un fluide parfait, on montre que le tenseur des contraintes a une expression particulièrement simple. Par conséquent, la modélisation des fluides parfaits se fait par des équations "minimales", c'est-à-dire avec le moins de termes possible. Si les domaines d'applications directes des fluides parfaits sont assez restreints (superfluides, aérodynamique), leur étude permet cependant de se concentrer sur la structure fondamentale des équations. Une partie du travail de modélisation consiste alors à intégrer ces structures dans des modèles plus complexes et à les concilier avec des termes additionnels modélisant des propriétés du fluide réel étudié, comme par exemple de la dissipation d'énergie

par frottement interne. Cette section est dédiée à la présentation des fluides parfaits et à la structure des équations qui les décrivent, avec une attention particulière portée sur le rôle joué par le tenseur de Reynolds. On introduit la notion de pression dans le paragraphe 2.1. Le paragraphe suivant 2.2 donne une brève présentation des systèmes hyperboliques de lois de conservation. Les paragraphes 2.1 et 2.2 contiennent des éléments standards et les personnes ayant déjà connaissance de ces sujets peuvent avancer directement aux paragraphes 2.3 et 2.4 présentant deux résultats obtenus pendant la thèse. Le paragraphe 2.3 introduit les équations moyennées avec tenseur de Reynolds. Le paragraphe 2.4 concerne le principe de Hamilton, avec une application aux écoulements multiphasiques.

2.1. Pression

2.1.1. Définition de la pression et équations d'Euler

Considérons un ensemble $\omega(t) \subset \Omega$ transporté par le fluide. Comme expliqué dans le paragraphe 1.3, cet ensemble est soumis à des forces surfaciques s'exerçant sur le bord $\partial\omega(t)$. Ces forces ont une densité σn , avec σ le tenseur des contraintes et n le vecteur normal sortant. On peut décomposer σn comme la somme de deux interactions : la résistance normale, qui est la projection de σn selon n , et la force de frottement, qui est la composante de σn tangente au bord $\partial\omega(t)$ (Figure 4).

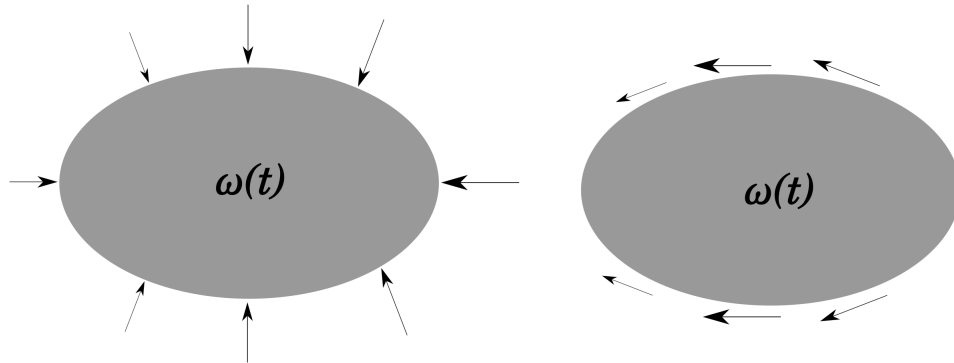


FIGURE 4. – Schéma des forces normales (à gauche) et tangentielles (à droite) s'exerçant sur $\partial\omega(t)$.

Définition 2. On appelle fluide parfait un fluide pour lequel les forces de frottement interne sont nulles, i.e. pour lequel σn est colinéaire à n en tout point. Par linéarité, il existe un scalaire $p \in \mathbb{R}$ tel que $\sigma = -p\text{Id}$. Ce scalaire est appelé la pression du fluide et représente une contrainte isotrope.

Remarque 11. Le signe “-” dans la définition de p est une convention pour que la pression soit positive lorsque la résistance normale s'exerce vers l'intérieur du volume considéré, comme c'est le cas dans la Figure 4 (par définition, la normale n est dirigée vers l'extérieur).

Pour un fluide parfait, l'équation d'évolution de la vitesse (0.37) s'écrit

$$\partial_t v + (v \cdot \nabla) v + \frac{1}{\rho} \nabla p = g. \quad (0.81)$$

Cette équation est appelée équation d'Euler. Sa forme conservative est

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + p \operatorname{Id})^T = \rho g. \quad (0.82)$$

Enfin, le bilan local d'énergie mécanique (0.54) devient

$$\partial_t \left(\frac{1}{2} \rho |v|^2 + \rho g z \right) + \operatorname{div} \left[\left(\frac{1}{2} \rho |v|^2 + \rho g z \right) v \right] = -v \cdot \nabla p. \quad (0.83)$$

Mathématiquement, il existe une distinction importante entre les modèles d'écoulements compressibles et incompressibles. En effet, les techniques d'analyse utilisées pour étudier ces modèles sont différentes, ainsi que les méthodes numériques pour leur résolution. L'importance de cette distinction peut être mise en évidence par le rôle joué par la pression dans chacun des cas.

— Dans le cas des équations d'Euler incompressibles :

$$\begin{cases} \operatorname{div}(v) = 0, \\ \partial_t \rho + v \cdot \nabla \rho = 0, \\ \partial_t v + (v \cdot \nabla) v + \frac{1}{\rho} \nabla p = g, \end{cases} \quad (0.84)$$

la contrainte d'incompressibilité $\operatorname{div}(v) = 0$ (voir définition 1) suffit à déterminer la pression à partir du champ de vitesse. En effet, en appliquant l'opérateur de divergence à l'équation d'évolution de la vitesse, on obtient une équation elliptique pour la pression :

$$-\operatorname{div} \left(\frac{1}{\rho} \nabla p \right) = \operatorname{div} [(v \cdot \nabla) v]. \quad (0.85)$$

On verra dans le paragraphe 2.4 que le principe d'action stationnaire de Hamilton permet d'interpréter la pression comme le multiplicateur de Lagrange associé à la contrainte d'incompressibilité.

Remarque 12. Dans le cas incompressible, l'identité $v \cdot \nabla p = \operatorname{div}(pv)$ permet de réécrire l'équation de l'énergie mécanique (0.54) comme une équation de conservation :

$$\partial_t \left(\frac{1}{2} \rho |v|^2 + \rho g z \right) + \operatorname{div} \left[\left(\frac{1}{2} \rho |v|^2 + \rho g z + p \right) v \right] = 0. \quad (0.86)$$

Pour un fluide parfait incompressible, l'énergie interne peut donc être prise égale à zéro.

— Dans le cas compressible, une équation supplémentaire est nécessaire pour obtenir un système d'équations fermé. On considère donc généralement une

relation liant la pression aux autres variables décrivant l'état du fluide (densité, température, salinité...). Cette relation est appelée une équation d'état. Ici, on se concentre sur les effets mécaniques, et on supposera donc que la pression dépend uniquement de la densité : $p = p(\rho)$. Un fluide pour lequel cette équation d'état est vérifiée est appelé barotrope. Les équations d'Euler compressible barotrope sont donc

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + (v \cdot \nabla) v + \frac{1}{\rho} \nabla p(\rho) = g. \end{cases} \quad (0.87)$$

Elles forment un système hyperbolique de lois de conservations, notion définie plus en détail dans le paragraphe 2.2. Ces équations admettent une loi de conservation supplémentaire : la conservation de l'énergie totale. Pour l'obtenir, on peut partir de l'équation d'évolution de l'énergie interne (0.60), qui s'écrit ici

$$\partial_t(\rho e) + \operatorname{div}(\rho e v) = \sigma : D(v) = -p(\rho) \operatorname{div}(v), \quad \text{i.e.} \quad \partial_t e + v \cdot \nabla e = -\frac{p(\rho)}{\rho} \operatorname{div}(v) \quad (0.88)$$

en utilisant la remarque 5. Or, en considérant une fonction $e : \mathbb{R}_+^* \rightarrow \mathbb{R}$ de classe C^1 et en multipliant l'équation de conservation de la masse (0.23) par $e'(\rho)$, on obtient formellement

$$\partial_t e(\rho) + v \cdot \nabla e(\rho) = -\rho e'(\rho) \operatorname{div}(v). \quad (0.89)$$

On voit que les équations (0.88) et (0.89) sont équivalentes pourvu qu'on ait

$$p(\rho) = \rho^2 e'(\rho). \quad (0.90)$$

L'équation (0.90) est appelée relation de Maxwell. Si $e(\rho)$ est défini par cette relation, on obtient donc l'équation de conservation de l'énergie totale :

$$\partial_t \left(\frac{1}{2} \rho |v|^2 + \rho g z + \rho e(\rho) \right) + \operatorname{div} \left[\left(\frac{1}{2} \rho |v|^2 + \rho g z + \rho e(\rho) + p(\rho) \right) v \right] = 0. \quad (0.91)$$

On donne au paragraphe 2.4 une interprétation de la conservation de l'énergie grâce au principe de Hamilton.

2.1.2. Pression hydrostatique et écoulements en couche mince

Une solution particulière des équations d'Euler est le fluide au repos, défini par $v = 0$. La densité est donc indépendante du temps, $\partial_t \rho = 0$, et les forces extérieures sont compensées par le gradient de pression uniquement. La pression est alors appelée pression hydrostatique. Dans le cas d'un fluide soumis uniquement à la gravité, on obtient

$$\nabla p = \rho g, \quad \text{d'où} \quad p(z_b) - p(z_a) = \int_{z_a}^{z_b} \rho g dz. \quad (0.92)$$

Notons que, pour obtenir cette dernière formule, il n'est pas nécessaire de supposer que le fluide est entièrement au repos. Il suffit en effet de considérer un fluide pour lequel l'accélération verticale est nulle.

Pour cette raison, la pression hydrostatique joue un rôle important dans le contexte des écoulements en couche mince. En effet, considérons les équations adimensionnées obtenues pour un écoulement incompressible en couche mince (0.70). Dans le cas d'un fluide parfait s'écoulant sur un plan horizontal ($\theta = 0$), on obtient

$$\begin{cases} \partial_{\tilde{x}} \tilde{u} + \partial_{\tilde{z}} \tilde{w} = 0, \\ F^2 [\partial_{\tilde{t}} \tilde{u} + \partial_{\tilde{x}}(\tilde{u}^2) + \partial_{\tilde{z}}(\tilde{u}\tilde{w})] = -\partial_{\tilde{x}} \tilde{p}, \\ \varepsilon^2 F^2 [\partial_{\tilde{t}} \tilde{w} + \partial_{\tilde{x}}(\tilde{u}\tilde{w}) + \partial_{\tilde{z}}(\tilde{w}^2)] = -1 - \partial_{\tilde{z}} \tilde{p}. \end{cases} \quad (0.93)$$

En utilisant la troisième équation et la condition aux limites $\tilde{p}(\tilde{h}) = 0$, on voit que la pression hydrostatique est une approximation de la pression à l'ordre ε^2 :

$$\tilde{p} = \tilde{h} - \tilde{z} + O(\varepsilon^2). \quad (0.94)$$

En négligeant les termes d'ordre ε^2 et en revenant dans les variables initiales, on obtient $p = \rho g(h - z)$. Les équations moyennées sur la profondeur de la masse et de la quantité de mouvement horizontale (0.79) s'écrivent alors

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x \left(hU^2 + \int_0^h (u - U)^2 dz + \frac{gh^2}{2} \right) = 0. \end{cases} \quad (0.95)$$

Les équations de Saint Venant (SAINT-VENANT 1871) sont données par

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x \left(hU^2 + \frac{gh^2}{2} \right) = 0. \end{cases} \quad (0.96)$$

Elles sont obtenues en retirant la contribution de l'intégrale de $(u - U)^2$ dans l'équation de la quantité de mouvement, ce qui revient à négliger les fluctuations de la vitesse. Mathématiquement, ces équations ont la même structure que les équations d'Euler compressible barotrope (0.87) en une dimension (sans accélération de la pesanteur dans le membre de droite). La hauteur h correspond à la densité ρ , et la vitesse moyenne U correspond à la vitesse v . L'équation d'état est ici $p(h) = gh^2/2$. Par conséquent, les équations de Saint Venant admettent également une loi de conservation de l'énergie :

$$\partial_t \left(\frac{hU^2}{2} + \frac{gh^2}{2} \right) + \partial_x \left[\left(\frac{hU^2}{2} + gh^2 \right) U \right] = 0. \quad (0.97)$$

Remarque 13. Dans le cas d'un écoulement tridimensionnel, on obtient des équations moyennées correspondant aux équations d'Euler compressible barotrope (0.87) avec

$d = 2$:

$$\begin{cases} \partial_t h + \operatorname{div}_h(h V_h) = 0, \\ \partial_t(h V_h) + \operatorname{div}_h\left(h V_h \otimes V_h + \frac{g h^2}{2} \operatorname{Id}\right)^T = 0. \end{cases} \quad (0.98)$$

L'équation de l'énergie est

$$\partial_t \left(\frac{h |V_h|^2}{2} + \frac{g h^2}{2} \right) + \operatorname{div}_h \left[\left(\frac{h |V_h|^2}{2} + g h^2 \right) V_h \right] = 0. \quad (0.99)$$

Cette correspondance montre que dans le contexte des écoulements en couche mince, les outils de la mécanique des fluides compressibles peuvent être utilisés pour étudier des modèles décrivant des écoulements incompressibles. Dans la section suivante, on donne une brève présentation du cadre mathématique adapté à l'étude des écoulements compressibles.

2.2. Outils mathématiques pour l'analyse des écoulements parfaits compressibles

Les équations d'Euler compressibles barotropes appartiennent à la famille des systèmes hyperboliques de lois de conservation. Comme son nom l'indique, cette famille d'équations aux dérivées partielles est définie par deux critères algébriques : hyperbolicité et caractère conservatif, qui sont détaillés dans les prochains paragraphes. Ces critères sont à la fois assez simples pour pouvoir être vérifiés dans des cas concrets, assez restrictifs pour garantir le caractère bien posé des équations, et assez généraux pour englober un grand nombre d'applications. En plus de leur utilité mathématique, ils ont également des interprétations physiques. L'hyperbolicité permet ainsi de définir des vitesses de propagation des ondes, et de garantir une propagation de l'information à vitesse finie. Le caractère conservatif est lié à la conservation des grandeurs physiques. La compréhension de ces critères permet enfin de construire des schémas numériques stables adaptés à ce type d'équations. Ce dernier point ne sera pas abordé en détail ici, on réfère donc aux livres de BOUCHUT [2004](#) ; GODLEWSKI et al. [2013](#) ; LEVEQUE [2002](#).

2.2.1. Hyperbolicité

L'objectif de ce paragraphe est de présenter certains concepts clefs concernant les équations hyperboliques. Par souci de simplicité, on se restreint au cas des systèmes à coefficients réels sur l'espace $\mathbb{R}^d$, mais la théorie a été aussi développée dans des cadres plus généraux. Deux références pour ce paragraphe sont les livres de BENZONI-GAVAGE et al. [2006](#) ; MÉTIVIER [2008](#). Considérons un système d'équations quasilinéaires donné par

$$\partial_t Y + \sum_{i=1}^d A_i(Y) \partial_i Y = f + B(Y), \quad Y|_{t=0} = Y_0, \quad (0.100)$$

avec $Y : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^n$ l'inconnue, $A_i : \mathbb{R}^n \rightarrow M_n(\mathbb{R})$ des matrices dépendant de Y de manière C^∞ , $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ de classe C^∞ tel que $B(0) = 0$, $\partial_i := \partial / \partial x_i$ ($i = 1, \dots, d$). Pour $Y \in \mathbb{R}^n$ et $\xi \in \mathbb{R}^d$, on définit

$$A(Y, \xi) := \sum_{i=1}^d A_i(Y) \xi_i \quad (0.101)$$

le symbole de l'équation (0.100). La définition suivante est donnée par MÉTIVIER 2008.

Définition 3. On dit que le système (0.100) est hyperbolique lorsqu'il existe une matrice $n \times n$ $S(Y, \xi)$, homogène de degré 0 en ξ , à coefficients C^∞ en (Y, ξ) pour $\xi \neq 0$ et telle que

- $S(Y, \xi)$ est symétrique et définie positive,
- Pour tout (Y, ξ) , $S(Y, \xi) A(Y, \xi)$ est symétrique.

$S(Y, \xi)$ est appelé un symétriseur symbolique associé au système (0.100).

Le théorème suivant énonce qu'un système hyperbolique est bien posé (MÉTIVIER 2008) :

Théorème 1. Soient $s > 1 + d/2$ et $T > 0$. On suppose que le système (0.100) est hyperbolique. Si $f \in C^0([0, T], H^s(\mathbb{R}^d))$ et $Y_0 \in H^s(\mathbb{R}^d)$, il existe $T' > 0$ et une unique solution $Y \in C^0([0, T'], H^s(\mathbb{R}^d))$ à (0.100).

Remarque 14. La condition $s > 1 + d/2$ est la condition minimale pour assurer que l'inconnue Y ainsi que ses dérivées spatiales soient continues en espace. Cela permet de définir sans ambiguïté le produit $A_i(Y) \partial_i Y$. On dit alors que Y est une solution forte. Pour une telle solution, le temps d'existence est local. En effet, du fait de la nonlinéarité des équations, des discontinuités peuvent apparaître en temps fini. Pour une solution $Y^* \in C^\infty$ par morceaux, le produit $A_i(Y^*) \partial_i Y^*$ n'est pas défini aux points de discontinuités de Y^* de manière univoque. En effet, $A_i(Y^*)$ est discontinu et $\partial_i Y^*$ est défini au sens des distributions grâce à une mesure singulière supportée par l'ensemble des points de discontinuités de Y^* . La notion de système de lois de conservation permet cependant de définir des solutions discontinues et d'étendre le temps d'existence au-delà de l'apparition de discontinuités.

Remarque 15. La définition de l'hyperbolicité ne dépend que du symbole $A(Y, \xi)$. Par conséquent, on se contente souvent d'étudier le système sans termes sources

$$\partial_t Y + \sum_{i=1}^d A_i(Y) \partial_i Y = 0, \quad Y|_{t=0} = Y_0. \quad (0.102)$$

On dit que le système (0.102) est homogène. Par linéarité du symbole, on peut également limiter l'étude au cas $|\xi| = 1$. Enfin, l'hyperbolicité est une notion invariante par changement lisse d'inconnue. En effet, si $Y = Y(Z)$, Z est solution de

$$\partial_t Z + \sum_{i=1}^d [\partial_Z Y^{-1} A_i(Y(Z)) \partial_Z Y] \partial_i Z = 0, \quad (0.103)$$

et un symétriseur symbolique est donné par $\tilde{S}(Z, \xi) = (\partial_Z Y)^T S(Y(Z), \xi) \partial_Z Y$.

En pratique, pour vérifier l'hyperbolicité d'un système quasilinéaire, on peut utiliser les critères suivants, qui sont plus restrictifs mais aussi plus simples à vérifier (et toujours invariants par changement d'inconnue) :

Critère 1. On dit que le système (0.100) est symétrisable (au sens de Friedrichs) s'il existe une matrice $n \times n$ $S(Y)$ de classe C^∞ telle que

- $S(Y)$ est symétrique définie positive,
- Pour tout (Y, ξ) , $S(Y)A(Y, \xi)$ est symétrique.

$S(Y)$ est appelé un symétriseur de (0.100). Dans le cas où on peut prendre $S(Y) = \text{Id}$, on dit que le système est symétrique.

Dans le cas d'un système symétrisable, la preuve du caractère bien-posé est plus simple et a été obtenue avant la preuve générale de Métivier (cf KATO 1975). L'idée de la preuve est de multiplier l'équation (0.100) par le symétriseur $S(Y)$, puis d'écrire une estimation d'énergie en utilisant la symétrie afin d'effectuer une méthode de point fixe (voir BENZONI-GAVAGE et al. 2006). Dans le cas général, il est nécessaire d'utiliser le calcul para-différentiel afin de pouvoir symétriser les équations à l'aide du symétriseur symbolique.

Critère 2. On dit que le système (0.100) est constamment hyperbolique si pour tout (Y, ξ) , le symbole $A(Y, \xi)$ est diagonalisable à valeurs propres réelles, et si les multiplicités des valeurs propres restent constantes, quelles que soient les valeurs de (Y, ξ) , $\xi \neq 0$.

Pour montrer qu'un système constamment hyperbolique admet un symétriseur symbolique, l'idée est d'écrire le symbole comme $A(Y, \xi) = Q(Y, \xi)D(Y, \xi)Q(Y, \xi)^{-1}$, avec $D(Y, \xi)$ une matrice diagonale et $Q(Y, \xi)$ une base de vecteurs propres. Le symétriseur symbolique peut alors être défini par $S(Y, \xi) = (Q(Y, \xi)^{-1})^T Q(Y, \xi)^{-1}$. Le fait d'avoir des multiplicités constantes empêche les valeurs propres de se croiser, ce qui permet de choisir $Q(Y, \xi)$ de sorte que $S(Y, \xi)$ soit de classe C^∞ (cf BENZONI-GAVAGE et al. 2006).

Remarque 16. Par le calcul précédent, on peut construire formellement un symétriseur symbolique dès que le symbole $A(Y, \xi)$ est diagonalisable à valeurs propres réelles. Cependant, la continuité du symétriseur symbolique ainsi obtenu n'est pas garantie en général. Un contre-exemple est présenté dans BENZONI-GAVAGE et al. 2006 dans le cas $n = 3$, $d = 2$ comme exemple de Petrowski, pour lequel le symbole $A(Y, \xi) = A_1 \xi_1 + A_2 \xi_2$ est diagonalisable à valeurs propres réelles, les matrices A_1 et A_2 sont constantes (avec des coefficients égaux à 0 ou 1) et le système correspondant est mal posé. Pour cette raison, la définition de l'hyperbolicité comme un système dont le symbole est diagonalisable à valeurs propres réelles, parfois utilisée dans la littérature, n'a pas été retenue ici. Notons que l'existence d'un symétriseur symbolique implique que le symbole est diagonalisable à valeurs propres réelles. En effet, si SA est symétrique et S est symétrique définie positive, alors

$$A = S^{-1}SA = S^{-1/2} [S^{-1/2}(SA)S^{-1/2}] S^{1/2} \quad (0.104)$$

et donc A est semblable à la matrice symétrique $S^{-1/2}(SA)S^{-1/2}$.

Exemple 8. Les équations d'Euler barotropes homogènes peuvent s'écrire sous la forme (0.100) avec

$$Y = \begin{pmatrix} \rho \\ v \end{pmatrix} \in \mathbb{R}^{d+1}, \quad A(Y, \xi) = \begin{pmatrix} v \cdot \xi & \rho \xi^T \\ \frac{p'(\rho)}{\rho} \xi & (v \cdot \xi) Id \end{pmatrix}. \quad (0.105)$$

Ces équations sont constamment hyperboliques dès que $p'(\rho) > 0$. Pour $|\xi| = 1$, les valeurs propres de $A(Y, \xi)$ sont $v \cdot \xi \pm \sqrt{p'(\rho)}$ (multiplicité 1) et $v \cdot \xi$ (multiplicité $d - 1$). Les équations sont également symétrisables : un symétriseur est donné par $S(Y) = \text{Diag}(p'(\rho)/\rho^2, 1, \dots, 1)$.

On peut interpréter physiquement les valeurs propres du symbole comme des vitesses de propagation d'ondes dans le système. En effet, si on linéarise le problème homogène (0.102) au voisinage d'une solution constante, notée $\bar{Y}$, on obtient l'équation suivante :

$$\partial_t Y + \sum_{i=1}^d A_i(\bar{Y}) \partial_i Y = 0. \quad (0.106)$$

En cherchant des solutions de la forme $Y = e^{i(k \cdot x - \omega t)}$, on obtient que la fréquence ω et le vecteur d'onde k sont liés par la relation de dispersion $\det[A(\bar{Y}, k) - \omega Id] = 0$. En particulier, si les valeurs propres du symbole $A(\bar{Y}, \xi)$ sont bornées sur la sphère $|\xi| = 1$, c'est aussi le cas des vitesses de phase $\omega/|k|$. L'information se propage donc à vitesse finie. Les schémas numériques préservant cette propriété ont alors l'avantage d'être facilement parallélisables.

Pour finir, on donne deux définitions supplémentaires en lien avec l'hyperbolicité.

Définition 4. On dit qu'un système quasilinéaire de la forme (0.100) est strictement hyperbolique si le symbole associé est diagonalisable avec des valeurs propres toutes réelles et de multiplicité 1.

Définition 5. On dit qu'un système quasilinéaire de la forme (0.100) est faiblement hyperbolique si le spectre du symbole est inclus dans $\mathbb{R}$.

2.2.2. Lois de conservation

Définition 6. On dit que le système (0.100) est sous forme conservative s'il existe des champs de vecteurs $F_1(Y), \dots, F_d(Y) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ tels que, pour $i = 1, \dots, d$, on ait $A_i(Y) = \partial_Y F_i(Y)$. L'équation (0.100) se réécrit alors

$$\partial_t Y + \sum_{i=1}^d \partial_i F_i(Y) = f + B(Y), \quad Y|_{t=0} = Y_0. \quad (0.107)$$

Les champs de vecteurs F_i correspondent au flux de la quantité conservée Y .

On considère désormais le cas homogène

$$\partial_t Y + \sum_{i=1}^d \partial_i F_i(Y) = 0. \quad (0.108)$$

Dans l'article FRIEDRICHS et al. 1971, Friedrichs et Lax montrent que, si le système (0.108) satisfait une loi de conservation supplémentaire, de type

$$\partial_t \eta(Y) + \operatorname{div}(j(Y)) = 0, \quad (0.109)$$

avec η une fonction strictement convexe, alors le système est symétrisable, et un symétriseur est donné par la matrice Hessienne de η . η est alors appelé une entropie du système, et j le flux d'entropie correspondant. Cela complète un résultat antérieur de GODUNOV 1961 qui montre que pour une certaine classe de système de lois de conservation dérivant de potentiels (incluant les équations d'Euler), une loi de conservation supplémentaire est automatiquement satisfaite et les équations peuvent être symétrisées. Dans le cadre des systèmes de lois de conservation, l'existence d'une entropie est donc un critère supplémentaire permettant de montrer la symétrisabilité (et donc l'hyperbolicité).

Exemple 9. Les équations d'Euler barotropes homogènes peuvent s'écrire comme un système de lois de conservation (0.108), avec

$$Y = \begin{pmatrix} \rho \\ \rho v \end{pmatrix} \in \mathbb{R}^{d+1}, \quad F(Y) = \begin{pmatrix} \rho v^T \\ \rho v \otimes v + p(\rho) \operatorname{Id} \end{pmatrix} \in M_{d+1,d}(\mathbb{R}) \quad (0.110)$$

et $F_i(Y)$ la i -ème colonne de $F(Y)$. Elles admettent comme équation de conservation supplémentaire la conservation de l'énergie totale :

$$\partial_t \left(\frac{1}{2} \rho |v|^2 + \rho e(\rho) \right) + \operatorname{div} \left[\left(\frac{1}{2} \rho |v|^2 + \rho e(\rho) + p(\rho) \right) v \right] = 0. \quad (0.111)$$

La densité d'énergie totale est une entropie pour les équations d'Euler. En effet, on a

$$\operatorname{Hess}(\eta)(Y) = \begin{pmatrix} \frac{|q|^2}{\rho^3} + 2e'(\rho) + \rho e''(\rho) & -\frac{q^T}{\rho^2} \\ -\frac{q}{\rho^2} & \frac{1}{\rho} \operatorname{Id} \end{pmatrix}, \quad \text{où } Y = \begin{pmatrix} \rho \\ q \end{pmatrix} \quad \text{et } \eta(Y) = \frac{|q|^2}{2\rho} + \rho e(\rho). \quad (0.112)$$

Par le critère de Sylvester, la matrice hessienne est définie positive si et seulement si son déterminant est positif, i.e. si et seulement si $2e'(\rho) + \rho e''(\rho) > 0$. Cela revient à dire que la densité volumique d'énergie interne $\rho \mapsto \rho e(\rho)$ est strictement convexe, ou, en utilisant la relation de Maxwell $p(\rho) = \rho^2 e'(\rho)$, que la pression est strictement croissante ($p'(\rho) > 0$). Le critère d'hyperbolicité $p'(\rho) > 0$ obtenu au paragraphe précédent peut donc être reformulé comme un critère de convexité de l'énergie totale.

Remarque 17. Pour un système de lois de conservation (0.107), on peut définir des solutions faibles. Ces solutions sont en général moins régulières que les solutions fortes, et peuvent admettre des discontinuités. En particulier, elles peuvent permettre de prolonger le temps d'existence d'une solution initialement régulière après l'apparition de discontinuités. Cependant, les solutions faibles ne sont pas uniques en général. Les paires entropies-flux d'entropie du système fournissent un critère de sélection pour ces solutions. On dit qu'une solution est entropique si elle satisfait au sens faible l'inéqua-

tion

$$\partial_t \eta(Y) + \operatorname{div}(j(Y)) \leq 0, \quad (0.113)$$

pour toute paire d'entropie-flux d'entropie (η, j) . Un théorème de KRUŽKOV 1970 (voir aussi SERRE 1999) assure l'existence et l'unicité des solutions entropiques pour une donnée initiale bornée dans le cas d'une équation scalaire ($n = 1$).

Les solutions faibles jouent également un rôle important dans la résolution numérique des systèmes de lois de conservation. Elles sont en effet au coeur de la méthode des volumes finis, proposée par Godunov (GODUNOV et BOHACHEVSKY 1959). Cette méthode consiste à approcher la solution recherchée par une fonction constante sur chaque maille à chaque pas de temps, et à résoudre un problème de Riemann (problème de Cauchy associé au système de lois de conservation avec une donnée initiale constante par morceaux) pour obtenir la solution au pas de temps suivant.

Pour toutes les raisons décrites ici, le cadre des systèmes hyperboliques de lois de conservation est bien adapté à la modélisation. Il permet en effet d'utiliser les propriétés venant de la physique pour garantir de bons comportements mathématiques pour les solutions. Comme illustré précédemment, ce cadre a beaucoup été développé dans le contexte des équations d'Euler compressibles. Dans la suite, on va utiliser ces éléments théoriques comme base d'analyse pour des modèles plus complexes incluant un tenseur de Reynolds, en lien avec les avalanches.

2.3. Tenseur de Reynolds

Les équations de Reynolds constituent un modèle déterministe de turbulence. De nombreux autres modèles existent et ne seront pas discutés ici ; on renvoie par exemple aux références MOHAMMADI et al. 1993 ; POPE 2001 ; WILCOX et al. 1998. Dans ce paragraphe, on montre comment dériver les équations de Reynolds dans le cas des écoulements moyennés sur la profondeur, puis on donne une liste des propriétés mathématiques de ces équations. Une de ces propriétés mathématiques (le résultat 1) constitue l'une des contributions de cette thèse.

2.3.1. Dérivation dans le cadre d'un écoulement moyenné

Reprenons les équations moyennées sur la profondeur pour un fluide parfait bidimensionnel s'écoulant sur un plan incliné horizontal (0.95) :

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x\left(hU^2 + \int_0^h (u - U)^2 dz + \frac{gh^2}{2}\right) = 0 \end{cases} \quad (0.114)$$

Ces équations sont obtenues en faisant l'hypothèse que la pression est hydrostatique, ce qui est vrai jusqu'à l'ordre ε^2 dans l'asymptotique des écoulements en couche

mince. On définit le tenseur de Reynolds, noté dans la suite P , par

$$P := \frac{1}{h} \int_0^h (u - U)^2 dz. \quad (0.115)$$

Il s'ensuit que $P \geq 0$, et $P = 0$ si et seulement si le profil de vitesse u est indépendant de z . Avec la condition $u(0) = 0$, on en déduit que $P > 0$, sauf à l'arrêt où $P = 0$. Le système (0.114) peut se réécrire

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x\left(hU^2 + hP + \frac{gh^2}{2}\right) = 0 \end{cases} \quad (0.116)$$

On obtient un système de deux équations et trois inconnues (h, U, P) . Une troisième équation est donc nécessaire pour décrire l'évolution de P . Pour obtenir cette équation, on va moyenner le bilan d'énergie (0.86)

$$\partial_t \left(\frac{1}{2} \rho |v|^2 + \rho g z \right) + \operatorname{div} \left[\left(\frac{1}{2} \rho |v|^2 + \rho g z + p \right) v \right] = 0. \quad (0.117)$$

Pour cela, on commence par réécrire ce bilan en négligeant les termes d'ordre ε^2 , i.e. la contribution de la vitesse verticale dans l'énergie cinétique et la partie non-hydrostatique de la pression. On obtient

$$\partial_t \left(\frac{1}{2} \rho u^2 \right) + \operatorname{div} \left[\left(\frac{1}{2} \rho u^2 + \rho g z + p_H \right) v \right] = 0, \quad (0.118)$$

avec $p_H = \rho g(h - z)$ la pression hydrostatique. On peut désormais intégrer cette équation sur la profondeur. Le calcul de l'équation moyennée se fait en deux étapes :

- Les termes faisant intervenir l'énergie cinétique sont calculés grâce au lemme 2 :

$$\int_0^h \left[\partial_t \left(\frac{1}{2} \rho u^2 \right) + \operatorname{div} \left(\frac{1}{2} \rho u^2 v \right) \right] dz = \partial_t \left(\int_0^h \frac{1}{2} \rho u^2 dz \right) + \partial_x \left(\int_0^h \frac{1}{2} \rho u^3 dz \right). \quad (0.119)$$

En utilisant la décomposition $u = u - U + U$ et la définition de P (0.115), on obtient

$$\int_0^h u^2 dz = hU^2 + hP \quad \text{et} \quad \int_0^h u^3 dz = hU^3 + 3hPU + \int_0^h (u - U)^3 dz. \quad (0.120)$$

- Il reste à calculer l'intégrale de $\operatorname{div}[(\rho g z + p_H)v]$. Pour cela, on voit tout d'abord que $\rho g z + p_H = \rho g h$. On va encore une fois utiliser le lemme 2 :

$$\int_0^h [\partial_t(gh) + \operatorname{div}(ghv)] dz = \partial_t \left(\int_0^h gh dz \right) + \partial_x \left(\int_0^h ghv dz \right) = \partial_t(gh^2) + \partial_x(gh^2U), \quad (0.121)$$

d'où

$$\int_0^h \operatorname{div}(ghv) dz = \partial_t (gh^2) + \partial_x (gh^2 U) - \int_0^h \partial_t (gh) dz = \partial_t \left(\frac{gh^2}{2} \right) + \partial_x (gh^2 U). \quad (0.122)$$

L'équation moyennée de l'énergie cinétique peut donc s'écrire

$$\partial_t \left(\frac{hU^2}{2} + \frac{hP}{2} + \frac{gh^2}{2} \right) + \partial_x \left(\frac{hU^3}{2} + \frac{3hPU}{2} + gh^2 U + \frac{1}{2} \int_0^h (u - U)^3 dz \right) = 0. \quad (0.123)$$

À l'aide des équations de conservation satisfaites par h et hU , on en déduit l'équation d'évolution de P :

$$\partial_t P + U \partial_x P + 2P \partial_x U + \frac{1}{h} \partial_x \left(\int_0^h (u - U)^3 dz \right) = 0. \quad (0.124)$$

Pour obtenir un système d'équations fermées sur les variables (h, U, P) , il faut donner une expression pour l'intégrale de $(u - U)^3$. Une méthode exacte serait de dériver une équation d'évolution pour cette quantité. On obtiendrait ainsi une quatrième équation, faisant intervenir l'intégrale de $(u - U)^4$. En continuant récursivement, on arriverait à un système infini d'équations couplées. Pour des raisons pratiques, cette méthode ne sera pas utilisée ici, et on se contentera des variables (h, U, P) . Pour simplifier, on fera ici l'hypothèse que le moment d'ordre 3 est négligeable, et on le prendra égal à zéro. Notons cependant que d'autres fermetures sont possibles. Par exemple, pour dériver le modèle d'avalanche présenté dans le paragraphe 3.2, le moment d'ordre 3 est exprimé de manière consistante comme un terme source en fonction des trois autres variables (voir chapitre 3). Ici, le modèle obtenu est donc

$$\begin{cases} \partial_t h + \partial_x (hU) = 0, \\ \partial_t (hU) + \partial_x \left(hU^2 + hP + \frac{gh^2}{2} \right) = 0, \\ \partial_t P + U \partial_x P + 2P \partial_x U = 0. \end{cases} \quad (0.125)$$

Ce modèle admet la conservation de l'énergie

$$\partial_t \left(\frac{hU^2}{2} + \frac{hP}{2} + \frac{gh^2}{2} \right) + \partial_x \left(\frac{hU^3}{2} + \frac{3hPU}{2} + gh^2 U \right) = 0. \quad (0.126)$$

Remarque 18. Dans le cas d'un écoulement tridimensionnel moyenné sur la profondeur, P est un tenseur d'ordre 2, défini par

$$P := \frac{1}{h} \int_0^h (v_h - V_h) \otimes (v_h - V_h) dz. \quad (0.127)$$

Il s'ensuit que P est un tenseur symétrique et positif. P est même défini positif, sauf dans le cas particulier où les profils de vitesse $u_x(z)$ et $u_y(z)$ sont égaux à une constante près.

Le système d'équations satisfaites par les variables (h, U, P) a été obtenu par Teshukov (TESHUKOV 2007) et s'écrit

$$\begin{cases} \partial_t h + \operatorname{div}_h(hV_h) = 0, \\ \partial_t(hV_h) + \operatorname{div}_h\left(hV_h \otimes V_h + hP + \frac{gh^2}{2}\operatorname{Id}\right)^T = 0, \\ \partial_t P + (V_h \cdot \nabla_h)P + \frac{\partial V_h}{\partial x_h}P + P\left(\frac{\partial V_h}{\partial x_h}\right)^T = 0. \end{cases} \quad (0.128)$$

Notons que dans ce dernier cas, trois équations scalaires sont nécessaires pour décrire l'évolution de P . Moyenner l'équation de l'énergie n'est donc pas suffisant pour déterminer l'évolution de P , il faut alors intégrer l'équation d'évolution de $u \otimes u$ (voir TESHUKOV 2007). Le système admet une équation de conservation de l'énergie totale, donnée par

$$\partial_t \left(\frac{h|V_h|^2}{2} + \frac{\operatorname{Tr}(hP)}{2} + \frac{gh^2}{2} \right) + \operatorname{div}_h \left[\left(\frac{h|V_h|^2}{2} + \frac{\operatorname{Tr}(hP)}{2} + hP + gh^2 \right) V_h \right] = 0. \quad (0.129)$$

2.3.2. Structure des équations

On présente ici une analyse de la structure des équations moyennées de Reynolds. Cette analyse est complétée par la dernière proposition du paragraphe (résultat 1), qui est un résultat original obtenu pendant la thèse. On considère donc le système d'équations quasilineaires donné par

$$\begin{cases} \partial_t h + \operatorname{div}(hV) = 0, \\ \partial_t(hV) + \operatorname{div}(hV \otimes V + hP + p(h)\operatorname{Id})^T = 0, \\ \partial_t P + (V \cdot \nabla)P + \frac{\partial V}{\partial x}P + P\left(\frac{\partial V}{\partial x}\right)^T = 0. \end{cases} \quad (0.130)$$

On se place en dimension $d = 1, 2$ ou 3 . En dimension $d = 1$ ou 2 , on retrouve les équations moyennées sur la profondeur obtenues par TESHUKOV 2007 en prenant $p(h) = gh^2/2$ (voir (0.125) et (0.128)). Il est également intéressant de considérer le cas $d = 3$ avec une loi de pression $p(h)$ générale car il apparaît dans le contexte des écoulements turbulents lorsqu'on effectue une moyenne de Reynolds. La variable h joue alors le rôle de la densité moyennée et est parfois notée $\bar{\rho}$. La vitesse V correspond à la vitesse de l'écoulement moyen, obtenue en prenant la moyenne de Favre de la vitesse instantanée. Le tenseur P est obtenu en prenant la moyenne de Favre des corrélations d'ordre 2 des fluctuations de vitesse. Ces équations dans le contexte de la turbulence sont présentées dans les références MOHAMMADI et al. 1993 ; POPE 2001 ; WILCOX et al. 1998. Pour cette raison, on considère ici le système (0.130), qui est un peu plus général que les équations obtenues précédemment, mais dont l'étude n'est pas plus complexe. Ce système admet la conservation de l'énergie totale

$$\partial_t(he_{tot}) + \operatorname{div}[(he_{tot} + p(h) + hP)V] = 0, \quad \text{avec} \quad e_{tot} := \frac{|V|^2}{2} + e(h) + \frac{\operatorname{Tr}P}{2}, \quad (0.131)$$

et $e(h)$ défini par la relation de Maxwell $h^2 e'(h) = p(h)$. Une autre loi de conservation est

$$\partial_t(h\psi) + \operatorname{div}(h\psi V) = 0, \quad \text{avec} \quad \psi := \frac{\det P}{h^2}. \quad (0.132)$$

On appelle ψ l'ensrophie, qui est reliée en dimension 1 au carré de la vorticité. Puisque l'ensrophie ψ est transportée, si initialement $\psi > 0$, alors cette inégalité reste vraie pour tout temps. De même, la symétrie du tenseur P est préservée par (0.130). On en déduit la proposition suivante :

Proposition 3. *Si initialement P est symétrique défini positif, alors P est symétrique défini positif pour tout temps.*

Cette proposition nous permet de restreindre l'étude au cas P symétrique défini positif. On a également une autre conséquence de la conservation de l'ensrophie et du fait que P est un scalaire en dimension 1 :

Proposition 4. *En dimension $d = 1$, le système (0.130) écrit dans les variables $(h, hV, h\psi)$ est un système de lois de conservations, qui admet en plus la conservation de l'énergie (0.131).*

Dans ces variables, la densité volumique d'énergie totale s'écrit $he_{tot} = hV^2/2 + he(h) + h^3\psi/2$. L'énergie est donc linéaire en $h\psi$, et n'est donc pas strictement convexe en ces variables. Cependant, l'énergie est une entropie pour le système écrit dans les variables conservatives $(h, hV, h\psi^{1/3})$. En dimension supérieure, on perd cette propriété, comme montré dans GAVRILYUK, IVANOVA et al. 2018 :

Proposition 5. *Les seules lois de conservations indépendantes satisfaites par le système (0.130) sont celles de conservation de la masse, de la quantité de mouvement, de l'énergie totale et de l'ensrophie. En particulier, si $d \geq 2$, les équations (0.130) ne peuvent pas être écrites comme un système de lois de conservation.*

Quand $d \geq 2$, on ne peut donc pas utiliser d'entropie pour montrer que les équations sont symétrisables, et donc hyperboliques. Il faut donc étudier directement le symbole du système (0.130). Concernant l'hyperbolicité, on peut énoncer le résultat suivant (voir BERTHON, COQUEL et al. 2002 ; TESHUKOV 2007) :

Proposition 6. *Les équations (0.130) sont constamment hyperboliques pour tout $h > 0$, $V \in \mathbb{R}^d$, $P = P^T > 0$ avec une loi de pression qui vérifie $p'(h) \geq 0$.*

Voici quelques éléments de preuve. Le système (0.130) peut être mis sous la forme quasilinéaire

$$\partial_t Y + \sum_{i=1}^d A_i(Y) \partial_i Y, \quad \text{avec} \quad Y = \begin{pmatrix} h \\ V \\ \tilde{P} \end{pmatrix} \in \mathbb{R}^{1+d+d(d+1)/2}. \quad (0.133)$$

Le symbole $A(Y, \xi)$ est donné pour $\xi \in \mathbb{R}^d$ par une matrice par blocs compatible avec Y :

$$A(Y, \xi) = \sum_{i=1}^d A_i(Y) \xi_i = \begin{pmatrix} V \cdot \xi & h \xi^T & 0 \\ \frac{1}{h} [P + p'(h)] \xi & (V \cdot \xi) \text{Id} & C(\xi) \\ 0 & D(\xi) & (V \cdot \xi) \text{Id} \end{pmatrix}. \quad (0.134)$$

En dimension 1, on a $\tilde{P} = P > 0$, $C(\xi) = \xi$, $D(\xi) = 2P\xi$. En dimension 2 et 3, on utilise la symétrie de P pour écrire $d(d+1)/2$ équations d'évolution pour les coefficients de P , notés $(P_{ij})_{1 \leq i, j \leq d}$, au lieu de d^2 équations. En dimension $d = 2$, on a

$$\tilde{P} = \begin{pmatrix} P_{11} \\ P_{12} \\ P_{22} \end{pmatrix}, \quad C(\xi) = \begin{pmatrix} \xi_1 & \xi_2 & 0 \\ 0 & \xi_1 & \xi_2 \end{pmatrix}, \quad D(\xi) = \begin{pmatrix} 2(P\xi)_1 & 0 \\ (P\xi)_2 & 2(P\xi)_1 \\ 0 & 2(P\xi)_2 \end{pmatrix}, \quad (0.135)$$

où $(P\xi)_i$ désigne le i -ème coefficient du vecteur $P\xi \in \mathbb{R}^d$. En dimension $d = 3$,

$$\tilde{P} = \begin{pmatrix} P_{11} \\ P_{12} \\ P_{13} \\ P_{22} \\ P_{23} \\ P_{33} \end{pmatrix}, \quad C(\xi) = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ 0 & \xi_1 & 0 & \xi_2 & \xi_3 & 0 \\ 0 & 0 & \xi_1 & 0 & \xi_2 & \xi_3 \end{pmatrix}, \quad (0.136)$$

et

$$D(\xi) = \begin{pmatrix} 2(P\xi)_1 & 0 & 0 \\ (P\xi)_2 & (P\xi)_1 & 0 \\ (P\xi)_3 & 0 & (P\xi)_1 \\ 0 & 2(P\xi)_2 & 0 \\ 0 & (P\xi)_3 & (P\xi)_2 \\ 0 & 0 & 2(P\xi)_3 \end{pmatrix}. \quad (0.137)$$

Pour $\xi \in \mathbb{R}^d$ avec $|\xi| = 1$, le polynôme caractéristique du symbole $A(Y, \xi)$ est donné par

$$\chi_{A(Y, \xi)}(\lambda) = [(V \cdot \xi - \lambda)^2 - 3\xi^T P \xi - p'(h)] [(V \cdot \xi - \lambda)^2 - \xi^T P \xi]^{d-1} (V \cdot \xi - \lambda)^{1+d(d-1)/2}. \quad (0.138)$$

Par conséquent, les valeurs propres sont données par

$$\begin{aligned} \lambda_{p\pm} &:= V \cdot \xi \pm \sqrt{3\xi^T P \xi + p'(h)}, \quad (\text{multiplicité } 1), \\ \lambda_{P\pm} &:= V \cdot \xi \pm \sqrt{\xi^T P \xi}, \quad (\text{multiplicité } d-1), \\ \lambda_V &:= V \cdot \xi \quad (\text{multiplicité } 1 + d(d-1)/2). \end{aligned}$$

Pour montrer que le symbole est diagonalisable, il faut exhiber une base de vecteurs propres correspondants. L'existence d'au moins un vecteur propre pour chaque valeur propre étant garantie, il suffit de traiter les valeurs propres $\lambda_{P\pm}$ et λ_V . Pour la valeur

propre $\lambda_{P\pm}$, les vecteurs propres s'écrivent

$$X_{P\pm} = \begin{pmatrix} 0 \\ \pm \sqrt{\xi^T P \xi} \xi^\perp \\ D(\xi) \xi^\perp \end{pmatrix}, \quad (0.139)$$

avec $\xi^\perp$ un vecteur orthogonal à ξ . Pour la valeur propre λ_V , un vecteur propre est donné par

$$X_V = \begin{pmatrix} -h \\ 0 \\ \alpha \end{pmatrix}, \quad (0.140)$$

avec $\alpha \in \mathbb{R}^{d(d+1)/2}$ tel que $C(\xi)\alpha = P\xi + p'(h)\xi$, et $d(d-1)/2$ autres vecteurs propres indépendants peuvent s'écrire

$$X_{V'} = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}, \quad (0.141)$$

où $\beta \in \mathbb{R}^{d(d+1)/2}$ tel que $C(\xi)\beta = 0$. En effet, $C(\xi) \in M_{d,d(d+1)/2}(\mathbb{R})$ est une matrice surjective, donc son noyau est de dimension $d(d-1)/2$ et les vecteurs X_V et $X_{V'}$ sont bien définis.

Pour compléter ces propriétés, on peut se demander si le système (0.130) est symétrisable au sens de Friedrichs en dimension 2 et 3. En effet, le caractère non conservatif des équations n'empêche pas la symétrisabilité a priori. Un premier résultat de cette thèse répond à cette question :

RÉSULTAT 1 (DELÉAGE 2024)

Soit $d = 2$ ou 3 . On suppose que $h > 0$ et que $P = P^T > 0$. Alors les deux propositions suivantes sont équivalentes :

1. le système (0.130) est symétrisable au sens de Friedrichs,
2. la pression est constante : $p'(h) = 0$, ou le tenseur P est une matrice scalaire, i.e. il existe $\lambda = \lambda(t, x) \in \mathbb{R}$ tel que $P = \lambda \text{Id}$.

Notons que, par ce qui précède, cette proposition est également vérifiée lorsque $d = 1$. Plus de détails concernant ce résultat ainsi qu'une preuve détaillée sont donnés dans le chapitre 1.

2.4. Principe de Hamilton

Dans les paragraphes précédents, on a présenté des éléments d'analyse de la structure mathématique de plusieurs systèmes d'équations différents (équations d'Euler compressibles barotropes, équations moyennées de Reynolds). Ces systèmes possèdent des propriétés communes : conservation de la masse, de la quantité de mouvement et de l'énergie totale, hyperbolicité. On peut se demander si ces similarités ont

une origine commune. Une réponse partielle est donnée par le principe de Hamilton, qui est présenté dans ce paragraphe. Le principe de l'action stationnaire de Hamilton, aussi appelé principe de Hamilton ou principe variationnel, est un outil théorique qui permet notamment de

- Présenter de manière unifiée un grand nombre de systèmes d'équations, dont les équations d'Euler et les équations de Reynolds,
- Dériver un système complet d'équations à partir d'une unique grandeur scalaire, appelée le Lagrangien,
- Garantir que le système ainsi obtenu satisfasse des lois de conservation adéquates.

Pour toutes ces raisons, le principe de Hamilton peut se révéler précieux pour la modélisation. Dans ce paragraphe, on donne une présentation de ce concept, puis on montre un exemple d'application dans le cadre des écoulements multiphasiques. Cet exemple d'application est un résultat original obtenu pendant la thèse.

2.4.1. Présentation du principe variationnel

On considère un milieu continu occupant un domaine $\Omega \subset \mathbb{R}^d$, dont le mouvement est décrit par des trajectoires $x(t, X) \in \Omega$, comme expliqué dans le paragraphe 1.1. La densité est définie comme précédemment dans les coordonnées lagrangiennes par

$$\rho^l J = \rho|_{t=0}, \quad \text{où } J = \det \frac{\partial x}{\partial X}, \quad (0.142)$$

(la notation f^l est introduite en 1.1), ce qui revient à imposer une équation de conservation pour la densité eulérienne ρ telle que $\rho(t, x(t, X)) = \rho^l(t, X)$. On définit alors la densité lagrangienne $\mathcal{L}$ par

$$\mathcal{L} := \frac{1}{2} \rho |v|^2 - \rho e_g, \quad (0.143)$$

avec e_g l'énergie potentielle généralisée, incluant la somme des énergies potentielle, interne et éventuellement d'autres contributions. Notons que la densité lagrangienne est donc la différence entre l'énergie cinétique et l'énergie potentielle généralisée, contrairement à l'énergie totale qui est la somme de ces deux quantités. Le Lagrangien associé est défini comme l'intégrale de la densité lagrangienne sur tout le domaine :

$$L = \int_{\Omega} \mathcal{L} dx \quad (0.144)$$

Finalement, l'action est définie comme l'intégrale du Lagrangien entre l'instant initial $t = 0$ et un instant final $T > 0$ fixé :

$$\mathcal{A} = \mathcal{A}[x] = \int_0^T L dt = \int_0^T \int_{\Omega} \mathcal{L} dx dt, \quad (0.145)$$

la notation $\mathcal{A}[x]$ signifiant ici que l'action est considérée comme une fonctionnelle dépendant de la trajectoire. Pour ε suffisamment proche de zéro et $\phi \in C_c^\infty([0, T] \times \Omega)^d$,

on considère une perturbation de la trajectoire x_ε donnée par

$$x_\varepsilon = x + \varepsilon \phi. \quad (0.146)$$

Notons que ϕ étant à support compact, l'état initial, l'état final et le comportement du milieu sur le bord $\partial\Omega$ ne sont pas perturbés. L'action correspondante est notée $\mathcal{A}_\varepsilon := \mathcal{A}[x + \varepsilon \phi]$. Le principe de l'action stationnaire de Hamilton énonce que les trajectoires empruntées par les particules sont celles pour lesquelles l'action est stationnaire, i.e. pour lesquelles on a, pour toute perturbation $\phi \in C_c^\infty[(0, T) \times \Omega]^d$,

$$\delta \mathcal{A} = 0, \quad (0.147)$$

où la variation δf est définie pour une quantité f_ε dépendant de ε par

$$\delta f := (\partial_\varepsilon f_\varepsilon)|_{\varepsilon=0}. \quad (0.148)$$

Exemple 10. *Équations d'Euler compressibles barotropes.*

Considérons la densité lagrangienne

$$\mathcal{L} = \mathcal{L}(\rho, v, z) = \frac{1}{2} \rho |v|^2 - \rho g z - \rho e(\rho). \quad (0.149)$$

La variation de l'action correspondante est donnée par

$$\delta \mathcal{A} = \int_0^T \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \rho} \delta \rho + \frac{\partial \mathcal{L}}{\partial v} \delta v \right) dx dt. \quad (0.150)$$

Il suffit donc de calculer les variations de la densité et de la vitesse pour calculer celle de l'action. Pour cela, on introduit ζ la variation de la trajectoire dans les coordonnées eulériennes, i.e. $\zeta^l = \phi = \delta x$. La relation $\rho_\varepsilon^l J_\varepsilon = \rho_0$ permet de calculer

$$\delta(\rho^l) = -\rho^l \frac{\delta J}{J} = -\rho^l [\operatorname{div}(\zeta)]^l, \quad (0.151)$$

d'où, en utilisant la correspondance

$$\delta(f^l) = (\delta f)^l + \delta x \cdot (\nabla f)^l, \quad (0.152)$$

on déduit que

$$\delta \rho = -\operatorname{div}(\rho \zeta). \quad (0.153)$$

De même, la relation $\delta(v^l) = \partial_t \delta x$ donne

$$\delta v + (\zeta \cdot \nabla) v = \partial_t \zeta + (v \cdot \nabla) \zeta, \quad \text{i.e.} \quad \delta v = \partial_t \zeta + (v \cdot \nabla) \zeta - (\zeta \cdot \nabla) v. \quad (0.154)$$

La variation de l'action peut donc s'écrire

$$\delta \mathcal{A} = \int_0^T \int_{\Omega} \left[\left(gz + e(\rho) + \rho e'(\rho) - \frac{1}{2} |v|^2 \right) \operatorname{div}(\rho \zeta) + \rho v \cdot (\partial_t \zeta + (v \cdot \nabla) \zeta - (\zeta \cdot \nabla) v) \right] dx dt. \quad (0.155)$$

Après intégration par parties, on obtient

$$\delta \mathcal{A} = - \int_0^T \int_{\Omega} [\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + p(\rho) Id)^T + \rho \nabla(gz)] \cdot \zeta dx dt. \quad (0.156)$$

La variation de l'action devant s'annuler pour tout ζ , on en déduit l'égalité

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + p(\rho) Id)^T = -\rho \nabla(gz). \quad (0.157)$$

On retrouve le bilan de quantité de mouvement des équations d'Euler barotrope. On dit que ce bilan est l'équation d'Euler-Lagrange associée au Lagrangien.

Remarque 19. Dans l'exemple précédent, l'équation de la quantité de mouvement joue un rôle différent de celui de l'équation de la masse. En effet, l'équation de la masse n'est pas une équation d'Euler-Lagrange, mais plutôt une contrainte qui est spécifiée dès le départ. L'équation (0.142) montre que cette contrainte peut être intégrée dans les coordonnées lagrangiennes. On parle alors de contrainte holonome.

Remarque 20. Dans le cas d'un fluide incompressible, la contrainte $\operatorname{div}(v) = 0$ est une contrainte holonome. En effet, elle s'écrit simplement dans les coordonnées lagrangiennes $J = 1$. Les équations d'Euler incompressibles peuvent alors être obtenues en appliquant le principe de Hamilton à la densité lagrangienne

$$\mathcal{L} = \frac{1}{2} \rho |v|^2 + p \left(1 - \frac{1}{K} \right) \quad (0.158)$$

avec K tel que $K^l = J$. La pression p est vue ici comme une variable indépendante de la trajectoire. L'équation d'Euler-Lagrange associée à cette variable donne la contrainte d'incompressibilité $\operatorname{div}(v) = 0$. On dit que la pression est le multiplicateur de Lagrange associé à cette contrainte.

Le théorème suivant, appelé théorème de Noether (NOETHER 1918), est un résultat fondamental qui permet de donner une interprétation supplémentaire aux lois de conservation d'un système obtenu par principe variationnel.

Théorème 2. À toute transformation infinitésimale qui laisse la densité lagrangienne invariante correspond une quantité conservée.

Exemple 11. On a les correspondances suivantes :

- La conservation de l'énergie totale $\rho |v|^2 / 2 + \rho e_g$ correspond à l'invariance du Lagrangien par translation temporelle et est vérifiée dès que la densité lagrangienne ne dépend pas explicitement du temps t .

- La conservation de la quantité de mouvement ρv correspond à l'invariance du Lagrangien par translation spatiale et est vérifiée dès que la densité lagrangienne ne dépend pas explicitement des coordonnées spatiales. Dans l'exemple précédent, la densité lagrangienne $\mathcal{L} = \rho|v|^2/2 - \rho g z - \rho e(\rho)$ dépend de la coordonnée verticale z , donc la composante verticale de la quantité de mouvement n'est pas conservée (il y a le terme source $-\rho g$ dans le membre de droite). En revanche, $\mathcal{L}$ ne dépend pas des coordonnées horizontales, donc la quantité de mouvement horizontale ρv_h est conservée.

Le résultat suivant de GAVRILYUK et GOUIN 2020 traite le cas des équations moyennées de Reynolds.

Proposition 7. *Le système (0.130) peut être obtenu par le principe de Hamilton. Plus précisément, l'équation de la quantité de mouvement*

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + p(\rho)\operatorname{Id} + \rho P)^T = 0 \quad (0.159)$$

est l'équation d'Euler-Lagrange associée à la densité lagrangienne

$$\mathcal{L} = \frac{1}{2}\rho|v|^2 - \rho e(\rho) - \frac{1}{2}\operatorname{Tr}(\rho P), \quad (0.160)$$

sous les contraintes de conservation de la masse

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad (0.161)$$

et d'évolution de P

$$\partial_t P + (v \cdot \nabla)P + \frac{\partial v}{\partial x}P + P\left(\frac{\partial v}{\partial x}\right)^T = 0. \quad (0.162)$$

Remarque 21. *Contrairement à la contrainte de conservation de la masse, la contrainte d'évolution de P n'est pas intégrable dans les coordonnées lagrangiennes. On parle alors de contrainte non holonome. Cependant, on peut exprimer la variation de P par l'équation (cf GAVRILYUK et GOUIN 2020)*

$$\delta P + (\zeta \cdot \nabla)P + \frac{\partial \zeta}{\partial x}P + P\left(\frac{\partial \zeta}{\partial x}\right)^T = 0. \quad (0.163)$$

C'est cette relation qui permet d'obtenir l'équation d'Euler-Lagrange.

2.4.2. Application à la modélisation des écoulements multiphasiques

Dans ce paragraphe, on présente un travail effectué au cours de cette thèse, qui est une application du principe de Hamilton à la modélisation des écoulements multiphasiques. Les écoulements multiphasiques jouent en effet un rôle important dans la modélisation des écoulements géophysiques (DREW et al. 2006). Dans le cas des avalanches de neige sèche, on utilise le concept d'écoulement granulaire i.e. le mouvement d'une phase solide composée d'un grand nombre de particules (la

neige) dans une phase gazeuse (l'air). Même si le rôle de l'air est souvent négligé pour modéliser la partie la plus dense de l'avalanche qui se trouve près du sol, il peut néanmoins être important pour la description de la couche de neige diluée qui se forme parfois dans la partie supérieure des avalanches de neige poudreuse (ANCEY 2001 ; ISSLER 2003, voir aussi KÖHLER et al. 2018 ; FAUG, TURNBULL et al. 2018 pour une classification plus complète des différents régimes d'avalanche).

On distingue deux grandes familles de modèles continus d'écoulements multiphasiques (DREW et al. 2006, voir aussi DRUI et al. 2019). Lorsque il est possible d'identifier des régions distinctes composées d'une seule phase interagissant les unes avec les autres, un modèle de phases séparées avec interfaces peut être utilisé. Lorsque la taille typique de ces régions "pures" est très petite devant la taille typique de l'écoulement, ou lorsque les phases sont très dispersées, il est plus adéquat d'utiliser une description moyennée. Dans ce dernier cas, on suppose que toutes les phases sont présentes en chaque point de l'espace, dans des proportions appelées fractions volumiques. C'est cette approche qui est utilisée ici.

On considère donc un mélange constitué de deux phases, notées phase 1 et phase 2, occupant un domaine $\Omega \subset \mathbb{R}^d$. Pour $i = 1, 2$, $t \geq 0$ et $\omega \subset \Omega$, on introduit $V_i(t, \omega)$ le volume occupé par la phase i dans l'ensemble ω . Il s'ensuit que $V_i(t, \cdot)$ est une mesure positive et absolument continue (i.e. $V_i(t, \omega) = 0$ si $|\omega| = 0$). Par conséquent, il existe une densité α_i appelée fraction de volume telle que, pour $\omega \subset \Omega$,

$$V_i(t, \omega) = \int_{\omega} \alpha_i(t, x) dx. \quad (0.164)$$

On suppose de plus que les deux phases occupent tout l'espace disponible (i.e. $V_1(t, \omega) + V_2(t, \omega) = |\omega|$) et que chacune des deux phases est présente en tout point de l'espace. En terme de fractions volumiques, cela s'écrit

$$\alpha_1 + \alpha_2 = 1, \quad \text{et} \quad 0 < \alpha_i < 1. \quad (0.165)$$

On voudrait obtenir des équations d'évolution pour les densités ρ_i et les vitesses v_i de chaque espèce, qui généraliseraient les équations d'Euler dans le cas d'un mélange biphasique. Pour cela, on va utiliser le principe variationnel décrit dans le paragraphe précédent. Un choix simple serait de considérer la densité lagrangienne suivante :

$$\mathcal{L} = \alpha_1 \mathcal{L}_1 + \alpha_2 \mathcal{L}_2 = \frac{1}{2} \rho_1 |v_1|^2 - \rho_1 e_1 \left(\frac{\rho_1}{\alpha_1} \right) + \frac{1}{2} \rho_2 |v_2|^2 - \rho_2 e_2 \left(\frac{\rho_2}{\alpha_2} \right), \quad (0.166)$$

avec e_i l'énergie interne de la phase i . Notons l'apparition du rapport $\rho_{i,0} := \rho_i / \alpha_i$, qui correspond à la densité de chaque espèce prise séparément.

Remarque 22. Dans le cas où une des deux phases, par exemple la phase 1, est constituée de particules solides, la densité matérielle correspondante $\rho_{1,0}$ doit être prise constante, et l'énergie interne e_1 égale à 0. Cela ne change pas la structure des équations.

En effet, l'expression (0.166) paraît justifiée par le fait que l'énergie est une quantité extensive. L'énergie cinétique du mélange est donc donnée par la somme des énergies

cinétiques de chaque phase, tout comme l'énergie potentielle généralisée. Cependant, le système obtenu en appliquant le principe variationnel à la densité lagrangienne (0.166) n'est pas toujours hyperbolique. Plus précisément, on peut montrer que pour tous v_1, v_2 avec $v_1 \neq v_2$ et

$$|v_2 - v_1|^2 < (\rho_1 + \rho_2) c_w^2 \left[\left(\frac{\alpha_1}{\rho_1} \right)^{1/3} + \left(\frac{\alpha_2}{\rho_2} \right)^{1/3} \right]^3 \quad (0.167)$$

avec c_w la vitesse du son dans le mélange, le symbole associé au système d'équations obtenu admet des valeurs propres de partie imaginaire non nulle (voir GAVRILYUK 2011). Par conséquent, la pertinence de ce modèle paraît questionnable.

Pour pallier à ce problème d'hyperbolicité, on considère dans cette thèse une densité lagrangienne plus complexe, donnée par

$$\mathcal{L} = \frac{1}{2} \rho_1 |v_1|^2 - \rho_1 e_1 \left(\frac{\rho_1}{\alpha_1} \right) + \frac{1}{2} \rho_2 |v_2|^2 - \rho_2 e_2 \left(\frac{\rho_2}{\alpha_2} \right) - \frac{1}{2} \text{Tr}(rP), \quad (0.168)$$

où P est un tenseur de Reynolds, dont l'évolution est prescrite par la contrainte

$$\partial_t P + (u \cdot \nabla) P + \frac{\partial u}{\partial x} P + P \left(\frac{\partial u}{\partial x} \right)^T = 0, \quad (0.169)$$

et où r représente une densité, et u une vitesse. Comme on considère un mélange, plusieurs choix sont possibles pour la densité r et la vitesse u associées à P . En effet, on peut prendre $r = \rho_1$, $r = \rho_2$ ou même $r = \rho$, avec ρ la densité du mélange définie par

$$\rho = \rho_1 + \rho_2. \quad (0.170)$$

De même, on peut choisir $u = v_1$, $u = v_2$ ou $u = v$, avec v la vitesse moyenne associée à la conservation de ρ :

$$\rho v := \rho_1 v_1 + \rho_2 v_2, \quad \text{de sorte que} \quad \partial_t \rho + \text{div}(\rho v) = 0. \quad (0.171)$$

Un premier résultat obtenu pendant cette thèse concernant la densité lagrangienne (0.168) est un critère simple permettant de choisir le couple (r, u) :

RÉSULTAT 2 (DELEAGE 2025)

Considérons le système d'équations obtenu par principe variationnel appliqué à la densité lagrangienne (0.168), sous les contraintes de conservation des densités ρ_1, ρ_2 et d'évolution de P (0.169). En dimension $d = 1$, toutes les valeurs propres du symbole associé sont réelles lorsque les vitesses v_1 et v_2 sont proches si, et seulement si, la densité r et la vitesse u sont "compatibles", i.e. liées par la relation

$$\partial_t r + \text{div}(r u) = 0. \quad (0.172)$$

Cette proposition est complétée par un second résultat, énoncé ici de manière

informelle :

RÉSULTAT 3 (DELEAGE 2025)

Considérons de nouveau le système (0.168)-(0.169), complété par la conservation des densités ρ_1, ρ_2 et la condition de compatibilité

$$\partial_t r + \operatorname{div}(r u) = 0. \quad (0.173)$$

Alors

- En dimension $d = 1$, le système associé est hyperbolique lorsque les vitesses v_1 et v_2 sont proches ;
- En dimension $d = 2, 3$, on peut garantir l'hyperbolicité lorsque v_1 et v_2 sont proches en imposant des conditions supplémentaires dites de non résonance. Si ces conditions ne sont pas garanties, le système est faiblement hyperbolique.

En particulier, la densité lagrangienne avec tenseur de Reynolds permet d'obtenir un système avec un bon comportement lorsque les deux vitesses sont proches, ce qui n'est pas le cas pour le système associé à la densité lagrangienne plus simple (0.166). On renvoie au chapitre 2 pour plus de contexte et de références, des énoncés plus précis et les preuves de chaque résultat. En particulier, on traite également dans cette section le cas supplémentaire d'une densité lagrangienne avec tenseur de Reynolds sur chacune des phases :

$$\mathcal{L} = \frac{1}{2}\rho_1|v_1|^2 - \rho_1 e_1 \left(\frac{\rho_1}{\alpha_1} \right) + \frac{1}{2}\rho_2|v_2|^2 - \rho_2 e_2 \left(\frac{\rho_2}{\alpha_2} \right) - \frac{1}{2}\operatorname{Tr}(\rho_1 P_1) - \frac{1}{2}\operatorname{Tr}(\rho_2 P_2). \quad (0.174)$$

Remarque 23. On a vu dans la section précédente que la même structure des équations moyennées de Reynolds apparaît dans deux contextes différents, tous deux liés à un opérateur de moyennisation. Le tenseur de Reynolds est alors défini par les corrélations d'ordre 2 des fluctuations de vitesses. En effet, c'est le cas dans le contexte des écoulements en couche mince, pour lesquels l'opérateur est donné par la moyennisation sur la profondeur, et dans le cas des écoulements turbulents, pour lesquels l'opérateur est la moyenne de Reynolds. Dans le contexte des écoulements multiphasiques, on peut également trouver des opérateurs de moyennisation. En effet, une autre manière de dériver un modèle de mélange à phases dispersées avec des fractions volumiques est de partir d'un modèle à phases séparées et d'appliquer un procédé de moyennisation spécifique (par exemple une moyenne d'ensemble, voir le livre de DREW et al. 2006). Il serait alors intéressant de savoir s'il est possible de dériver dans ce contexte une équation d'évolution pour les secondes corrélations de la vitesse microscopique, définies à partir de l'opérateur de moyennisation spécifique, qui serait analogue à l'équation d'évolution du tenseur de Reynolds présentée ici.

Dans la section suivante, on quitte le domaine des fluides parfaits afin de pouvoir modéliser les effets de friction qui sont prédominants dans la partie dense des ava-

lanches granulaires. Dans ce régime, le rôle joué par l'air environnant peut souvent être négligé, car la densité de l'air est très inférieure à celle des particules solides. Considérer un modèle monophasé est alors une bonne approximation qui permet de simplifier drastiquement les équations (ISSLER 2003). Pour dériver les équations, on n'utilisera pas le principe variationnel, mais on se basera sur des lois empiriques obtenues expérimentalement. En effet, une des limites du principe de Hamilton est qu'il procure des modèles pour lesquels l'énergie est forcément conservée, grâce au théorème de Noether. Or les phénomènes de friction, étant opposés au mouvement, sont généralement associés à des dissipations d'énergie, et doivent donc être modélisés autrement.

3. Rhéologies granulaires

Cette section a pour but de présenter quelques propriétés des écoulements granulaires denses ainsi que certaines équations utilisées pour les modéliser. Une attention particulière est portée aux écoulements gravitaires sur plan incliné, puisqu'on s'intéresse in fine aux phénomènes d'avalanches. Dans ce contexte, le comportement du milieu peut être décrit par une loi de friction. On commence donc par introduire les lois de frictions dans le paragraphe 3.1, d'abord à l'échelle macroscopique, puis à des échelles de plus en plus réduites, jusqu'à la description d'une rhéologie locale (rhéologie $\mu(I)$ incompressible) au paragraphe 3.2. Cela permet alors de présenter un résultat de la thèse, qui est la dérivation et l'analyse d'un modèle moyenné sur la profondeur consistant avec la rhéologie $\mu(I)$. On donne ensuite en 3.3 un aperçu des différents modèles présentés dans la littérature pour généraliser cette rhéologie aux écoulements compressibles. On en profite pour énoncer un résultat supplémentaire de cette thèse, qui est la dérivation d'un modèle moyenné sur la profondeur consistant avec une extension compressible de la rhéologie $\mu(I)$. La fin de la section (paragraphe 3.4) est une introduction au cas des suspensions (phase granulaire immergée dans un liquide). On présente dans ce contexte un autre résultat de la thèse : la stabilité asymptotique d'une onde progressive pour un modèle jouet de suspension.

3.1. Lois de friction

Les lois de friction permettent de décrire le comportement des forces d'interaction entre deux solides en contact. Un coefficient de friction est utilisé afin de relier les forces normale et tangentielle. On commence par présenter la loi de friction la plus simple (loi de Coulomb), puis on introduit une loi plus complexe obtenue expérimentalement par POULIQUEN 1999b dans le cas d'un écoulement granulaire sur fond rugueux.

3.1.1. Loi de Coulomb

La loi de Coulomb peut s'énoncer facilement à l'échelle macroscopique. Considérons en effet un objet de masse m en contact avec un second objet. Les forces de

contact peuvent alors se décomposer en une résistance normale R_N , i.e. une composante perpendiculaire à la surface de contact, et une résistance tangentielle R_T (voir Figure 5). La loi de Coulomb peut alors être énoncée ainsi :

- Il existe un coefficient μ_s (coefficient de friction statique) tel que, en cas de non glissement, $|R_T| \leq \mu_s |R_N|$, et
- Il existe un coefficient μ_d (coefficient de friction dynamique) tel que, en cas de glissement, R_T est opposé à la vitesse de glissement, avec $|R_T| = \mu_d |R_N|$.

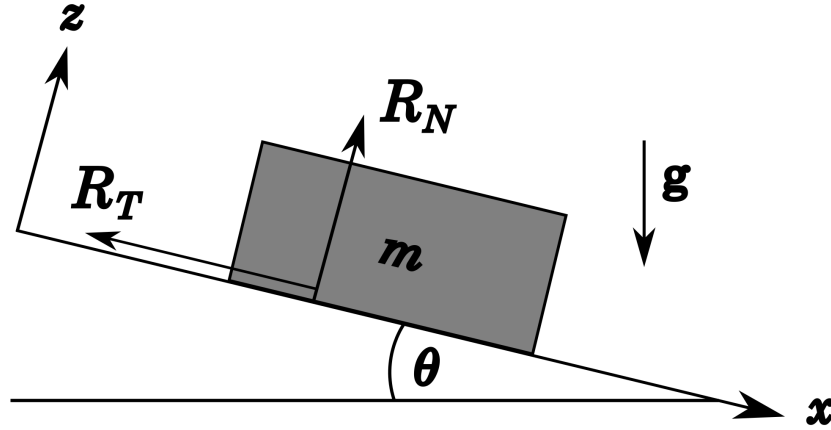


FIGURE 5. – Schéma de solide sur plan incliné

Le cas qui nous intéresse ici est celui d'une masse soumise à la gravité sur un plan incliné faisant un angle θ avec l'horizontale (Figure 5). Dans ce cas, la condition de contact permet de déterminer $|R_N| = mg \cos \theta$. Le principe fondamental de la dynamique projeté sur l'axe horizontal s'écrit alors

$$m\ddot{x} = mg \sin \theta + R_T. \quad (0.175)$$

On en déduit le comportement suivant :

- Si l'objet est à l'arrêt, alors $R_T = -mg \sin \theta$, et donc $\tan \theta \leq \mu_s$. Le coefficient de frottement statique peut donc s'interpréter comme la pente à partir de laquelle une masse initialement immobile commence à glisser.
- Si l'objet est en mouvement (dans le sens de la descente), on a $R_T = -\mu_d mg \cos \theta$. L'accélération s'écrit donc $\ddot{x} = g \cos \theta (\tan \theta - \mu_d)$. Trois comportements sont possibles :
 1. Si $\tan \theta < \mu_d$, l'objet ralentit jusqu'à s'arrêter. On retrouve alors le cas précédent, et donc $\tan \theta \leq \mu_s$. Par conséquent, on en déduit la relation $\mu_d \leq \mu_s$.
 2. Si $\tan \theta = \mu_d$, l'objet descend à vitesse constante.
 3. Si $\tan \theta > \mu_d$, la descente est uniformément accélérée.

Par conséquent, on voit que le coefficient dynamique peut s'interpréter comme la pente d'équilibre à laquelle un mouvement stationnaire est possible.

La loi de Coulomb a été utilisée à l'échelle des écoulements moyennés sur la profondeur par Savage et Hutter (SAVAGE et al. 1989). Rappelons que les équations exactes sont données par (0.79) :

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x\left(hU^2 + \int_0^h (u-U)^2 dz\right) = gh \sin \theta + \frac{1}{\rho} \left[\partial_x \left(\int_0^h \sigma_{xx} dz \right) - \sigma_{xz}(0) \right]. \end{cases} \quad (0.176)$$

Pour obtenir un système d'équations fermé, Savage et Hutter font les hypothèses suivantes :

- L'intégrale de u^2 est approchée par

$$hU^2 + \int_0^h (u-U)^2 dz = \alpha hU^2, \quad (0.177)$$

où $\alpha \geq 1$ est un paramètre appelé facteur de forme.

Remarque 24. Si $\alpha = 1$, on retrouve l'hypothèse utilisée pour dériver le modèle de Saint Venant (0.96) consistant à négliger les fluctuations de vitesses. Si $\alpha > 1$, ces fluctuations sont prises en compte. Notons que dans le cas $\alpha > 1$, l'invariance galiléenne des équations et l'équation de l'énergie totale ne sont plus vérifiées, comme montré par RICHARD et GAVRILYUK 2012. De plus, aucune raison ne justifie a priori le fait que α soit constant dans tout l'écoulement. En ce sens, le modèle augmenté avec tenseur de Reynolds présenté dans le paragraphe 2.3 permet de résoudre ces contradictions, puisqu'il revient à considérer un coefficient α variable qui possède sa propre équation d'évolution, compatible avec un bilan d'énergie.

- Le terme de frottement au fond $-\sigma_{xz}(0)$, qui correspond à la résistance tangentielle du support, est exprimé par la loi de Coulomb : $|\sigma_{xz}(0)| = \mu |\sigma_{zz}(0)|$. Le coefficient $\sigma_{zz}(0)$ (la résistance normale) est donné en faisant l'approximation hydrostatique $\sigma_{zz} = \rho g(z-h) \cos \theta$, qui est vraie en négligeant les termes d'ordre ε dans les variables adimensionnées (voir équation (0.70)). On en déduit que $\sigma_{xz}(0) = \mu \operatorname{sgn}(U) \rho g h \cos \theta$.
- L'intégrale de σ_{xx} est calculée en faisant l'hypothèse que $\sigma_{xx} = K \sigma_{zz}$. K est un facteur de proportionnalité qui est donné explicitement par Savage et Hutter grâce à un critère de Mohr-Coulomb, inspiré de la mécanique des sols mais appliqué ici à un milieu granulaire qui s'écoule. Si $K = 1$, les contraintes normales sont isotropes et peuvent être interprétées comme une pression, qui est ici hydrostatique ($\sigma_{xx} = \sigma_{zz} = -p_H$). Si $K \neq 1$, on dit que les contraintes normales sont anisotropes.

Finalement, le modèle de Savage et Hutter peut s'écrire ainsi :

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x\left(\alpha hU^2 + \frac{Kgh^2 \cos \theta}{2}\right) = gh \cos \theta [\tan \theta - \mu \operatorname{sgn}(U)]. \end{cases} \quad (0.178)$$

Notons que, comme pour le cas macroscopique, l'équilibre stationnaire uniforme n'est possible que pour $\tan \theta = \mu$. Pour $\alpha = 1$, on retrouve la structure des équations de Saint Venant avec un terme source.

3.1.2. Loi de friction de Pouliquen

À partir du modèle pionnier de Savage et Hutter, d'autres modèles ont été dérivés afin de prendre en compte des effets supplémentaires. Concernant la loi de friction, un développement important a été effectué par POULIQUEN 1999b. Le dispositif expérimental considéré est constitué d'un plan inclinable sur lequel une couche de grains a été collée afin d'obtenir un fond rugueux et de forcer au mieux la condition de non glissement. En amont du plan incliné, un réservoir avec une porte de hauteur réglable permet d'obtenir des écoulements granulaires dont l'épaisseur est contrôlée. Les particules utilisées pour l'expérience sont des billes de verre (particules sphériques). Grâce à ce dispositif, des écoulements stationnaires et uniformes sont obtenus. La mesure des paramètres de ces écoulements permet alors d'établir une nouvelle loi de friction plus précise, donnant de bonnes prédictions théoriques dans plusieurs configurations différentes (voir la fin de ce paragraphe). On résume ci-dessous des résultats importants de l'article de Pouliquen dans l'établissement de cette loi de friction, qui sont illustrés par la Figure 6.

- Des écoulements stationnaires et uniformes sont obtenus pour le même matériau avec des pentes différentes. En particulier, ces écoulements ne peuvent pas être décrits par le modèle de Savage et Hutter (0.178).
- À chaque inclinaison θ correspond une hauteur minimale en-dessous de laquelle aucun écoulement stationnaire uniforme n'est possible, et les particules s'arrêtent. Cette hauteur minimale est notée $h_{stop}(\theta)$. Une expression analytique pour $h_{stop}(\theta)$ correspondant aux mesures expérimentales est donnée par Pouliquen :

$$\tan \theta = \mu_1 + (\mu_2 - \mu_1) \exp\left(-\frac{h_{stop}}{L}\right). \quad (0.179)$$

Le paramètre μ_1 correspond à la pente en-dessous de laquelle tous les écoulements s'arrêtent ($h_{stop}(\theta) \rightarrow \infty$ lorsque $\tan \theta \rightarrow \mu_1$). Le paramètre μ_2 est la pente à partir de laquelle $h_{stop} = 0$, i.e. aucun grain ne peut s'arrêter. Le paramètre L est une longueur typique. D'autres formules pour h_{stop} ont été proposées, notamment (voir POULIQUEN et FORTERRE 2002)

$$\tan \theta = \mu_1 + \frac{\mu_2 - \mu_1}{1 + h_{stop}/L}. \quad (0.180)$$

- La vitesse U de l'écoulement, sa hauteur h et l'angle θ vérifient à l'équilibre la relation

$$F = \beta \frac{h}{h_{stop}(\theta)}, \quad \text{avec} \quad F = \frac{|U|}{\sqrt{gh}} \quad (0.181)$$

le nombre de Froude (voir (0.68)) et $\beta > 0$ un paramètre indépendant de l'in-

clinaison, de la taille des grains et de la rugosité. Notons que cette relation est parfois écrite en remplaçant F par $\mathcal{F} := F/\sqrt{\cos\theta}$, ce qui permet de prendre en compte la pente dans la définition du nombre de Froude.

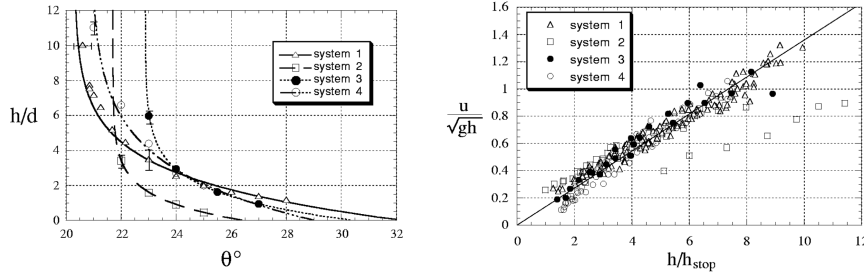


FIGURE 6. – Résultats expérimentaux de POULIQUEN 1999b. À gauche, mesures et expressions analytique de $h_{stop}(\theta)$ pour les différents systèmes de grains étudiés. À droite, corrélation entre le nombre de Froude et le rapport h/h_{stop} .

À partir de ces considérations, il est possible de déduire une loi de friction pour les écoulements granulaires. En effet, on a vu précédemment que l'équilibre est atteint lorsque le frottement est égal à la pente : $\mu = \tan\theta$. Pour avoir des écoulements stationnaires et uniformes à des pentes différentes, il faut donc considérer un coefficient de frottement μ variable. À l'équilibre, on peut exprimer $\mu = \tan\theta$ en fonction de h_{stop} par la relation (0.179) (ou (0.180)). De même, on peut exprimer h_{stop} en fonction de h et U via la relation (0.181). En combinant ces équations, on obtient une expression de μ en fonction de h et de U qui est valable à l'équilibre. L'hypothèse de Pouliquen est que la relation $\mu = \mu(h, U)$ reste vraie même pour des écoulements hors équilibre. La loi de friction proposée s'écrit

$$\mu = \mu(h, U) = \tan\theta_{stop} \left(\frac{\beta h \sqrt{gh}}{|U|} \right), \quad (0.182)$$

où $\theta_{stop}(h)$ est la fonction réciproque de $h_{stop}(\theta)$. Le système d'équations moyennées sur la profondeur correspondant est donc

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x \left(\alpha h U^2 + \frac{Kgh^2 \cos\theta}{2} \right) = gh \cos\theta [\tan\theta - \mu(h, U) \text{sgn}(U)]. \end{cases} \quad (0.183)$$

Remarque 25. Rien ne garantit a priori que la relation $\mu = \mu(h, U)$, obtenue pour des écoulements stationnaires et uniformes, reste valide pour des écoulements non-stationnaires et/ou non-uniformes. Les travaux cités ci-dessous montrent cependant que cette hypothèse permet d'obtenir des résultats satisfaisants dans plusieurs configurations hors équilibre. Une contribution de cette thèse, présentée au paragraphe 3.2.3, est une estimation théorique de l'écart entre la loi de friction (0.182) et la friction au sein d'un

écoulement non stationnaire uniforme, ainsi que la prise en compte de cet écart dans un modèle moyenné sur la profondeur (voir résultats 4 et 5).

Le modèle (0.183) a été utilisé dans de nombreuses situations. Citons ici trois exemples d'applications (en plus de la possibilité de prédire des écoulements stationnaires uniformes sur des pentes de valeurs différentes).

- Ce système a été utilisé pour prédire le profil d'un front granulaire sec (POULIQUEN 1999a, voir Figure 7).
- Le modèle a également été utilisé pour prédire l'apparition d'ondes de surfaces (FORTERRE et POULIQUEN 2003). Les ondes de surface peuvent être observées lorsque les effets de la gravité ne sont pas suffisants pour compenser l'instabilité causée par les termes inertiels. Mathématiquement, cela peut se caractériser par l'existence d'un nombre de Froude critique $\mathcal{F}_c$. L'instabilité a lieu si et seulement si $\mathcal{F} > \mathcal{F}_c$. Le modèle (0.183) prédit, lorsque $K = 1$ et $\alpha = 1$, un nombre de Froude critique $\mathcal{F}_c = 2/3$. Ce résultat est proche de la valeur mesurée expérimentalement par FORTERRE et POULIQUEN 2003 (entre 0.53 et 0.63 selon la pente). Le modèle ne permet cependant pas de prédire le taux de croissance de cette instabilité lorsque la fréquence de la perturbation augmente. En effet, Forterre et Pouliquen observent expérimentalement une fréquence de coupure, au-delà de laquelle le taux de croissance devient négatif et l'instabilité disparaît, alors que le modèle prédit un taux de croissance positif quelle que soit la fréquence.
- Une version bidimensionnelle a été utilisée par POULIQUEN et FORTERRE 2002 pour prédire l'étalement d'une masse de grains. Pour prendre en compte l'arrêt, la loi de friction $\mu = \mu(h, U)$, qui peut s'interpréter comme un coefficient de friction dynamique variable, est complétée grâce à un coefficient de friction statique $\mu_s = \mu_s(h)$, également déterminé expérimentalement. La loi de friction utilisée est alors une interpolation entre les coefficients dynamique et statique. La transition du régime dynamique vers le régime statique se fait lorsque le nombre de Froude F devient inférieur à β . Par la relation (0.181), cela correspond à l'équilibre à $h < h_{stop}$. Lorsque h devient inférieur à h_{stop} , l'augmentation de la friction permet d'arrêter l'écoulement, ce qui est conforme aux observations de POULIQUEN 1999b. Le choix du critère de transition $F < \beta$ au lieu de $h < h_{stop}$ permet d'avoir une loi de friction bien définie pour n'importe quelle pente, alors que la fonction $h_{stop}(\theta)$ n'est définie que pour $\mu_1 < \tan \theta < \mu_2$. Des lois de friction similaires avec d'autres critères de transition entre le régime statique et le régime dynamique ont été proposés par EDWARDS, VIROULET et al. 2017 et EDWARDS, RUSSELL et al. 2019.

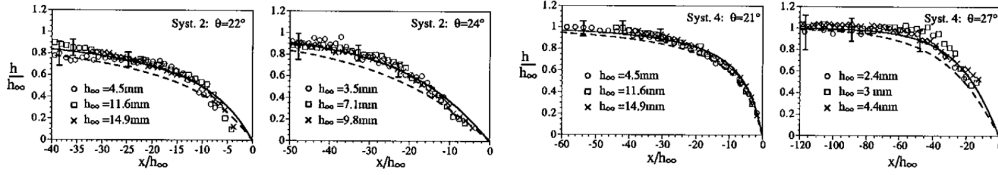


FIGURE 7. – Profils de fronts granulaires secs. En trait plein, prédiction du modèle (0.183). Les points sont les mesures expérimentales effectuées par POULIQUEN 1999a.

3.2. Rhéologie frictionnelle

Dans ce paragraphe, on montre comment les lois de frictions décrites précédemment peuvent être utilisées pour obtenir des rhéologies frictionnelles locales. Plus précisément, on se concentrera sur la rhéologie $\mu(I)$. Après avoir décrit les caractéristiques principales de cette rhéologie, on montre comment elle peut être utilisée pour affiner les modèles moyennés sur la profondeur. En particulier, on présente un modèle dérivé pendant cette thèse à partir de cette rhéologie.

3.2.1. Hypothèses effectuées dans l'écriture de la rhéologie

La rhéologie $\mu(I)$, proposée par JOP et al. 2006, peut être énoncée comme une transposition à l'échelle mésoscopique de la loi de friction $\mu = \mu(h, U)$ obtenue par POULIQUEN 1999b et décrite dans le paragraphe précédent. Elle possède en effet les deux caractéristiques suivantes :

- La contrainte normale et la contrainte tangentielle sont proportionnelles. Le coefficient de proportionnalité ou coefficient de friction est noté μ .
- Le coefficient de friction μ peut prendre plusieurs valeurs en fonction des paramètres de l'écoulement.

Pour obtenir une rhéologie locale, Jop, Forterre et Pouliquen font plusieurs hypothèses supplémentaires. La première hypothèse est que l'écoulement est incompressible, avec une densité constante. Comme expliqué dans le paragraphe 1.3.2, à l'échelle mésoscopique, les contraintes internes sont décrites par le tenseur des contraintes σ . La distinction entre les contraintes normales et tangentielles peut alors être effectuée en décomposant le tenseur des contraintes σ en deux tenseurs orthogonaux :

$$\sigma = -p\text{Id} + \tau. \quad (0.184)$$

La pression p est le multiplicateur de Lagrange associé à la contrainte d'incompressibilité. Comme expliqué dans le paragraphe 2.1, la matrice $-p\text{Id}$ décrit une contrainte normale isotrope. On suppose que les matrices $-p\text{Id}$ et τ sont orthogonales, ce qui s'écrit simplement $\text{Tr}(\tau) = 0$. τ est appelé le déviateur et sert à modéliser les contraintes tangentielles. En particulier, la contrainte totale $|\sigma|$ s'exprime grâce aux contraintes normales et tangentielles : $|\sigma|^2 = p^2 n/2 + |\tau|^2$ (avec n la dimension d'espace). La proportionnalité entre les contraintes normales et tangentielles s'écrit simplement

$|\tau| = \mu p$. Il faut ensuite spécifier la direction du tenseur τ . Pour cela, Jop, Forterre et Pouliquen font l'hypothèse que τ est aligné avec le tenseur des taux de déformation $\dot{\gamma}$, défini par

$$\dot{\gamma} = \nabla v + (\nabla v)^T = 2D(v). \quad (0.185)$$

Notons que $\dot{\gamma}$ est un tenseur symétrique, de trace nulle puisque l'écoulement est supposé incompressible. Lorsque $\dot{\gamma} = 0$, il n'y a pas de déformation (voir le lemme 1). À l'échelle macroscopique, la friction est opposée à la vitesse de glissement, i.e. à la différence de vitesse entre les deux objets en contact. À l'échelle locale, l'alignement de τ avec $\dot{\gamma}$ signifie que la friction est opposée à la déformation.

Remarque 26. *L'alignement du déviateur τ avec le tenseur des déformations $\dot{\gamma}$ peut également s'interpréter en terme de dissipation d'énergie. À l'échelle macroscopique, la dissipation d'énergie cinétique par unité de temps causée par le frottement est donnée par la puissance $\mathcal{P}$ de cette force, i.e. par le produit scalaire entre cette force et la vitesse de l'objet. Lorsqu'il y a contact, la vitesse est uniquement tangentielle, et donc le travail de la résistance normale est nul. De plus, le fait que la résistance tangentielle soit opposée à la vitesse de glissement v permet d'écrire la puissance comme*

$$\mathcal{P} = -|R_T||v|. \quad (0.186)$$

L'alignement de la résistance tangentielle et de la vitesse de glissement peut donc s'interpréter ainsi : la direction de la résistance tangentielle maximise la dissipation d'énergie liée au frottement. À l'échelle mésoscopique, la puissance des frottements internes s'écrit (voir section 1.4) :

$$\mathcal{P} = - \int \sigma : D(v) = - \int \tau : D(v) = - \int \mu(I) p \frac{\tau}{|\tau|} : D(v), \quad (0.187)$$

en utilisant la contrainte d'incompressibilité et la relation $|\tau| = \mu(I) p$. Par conséquent, lorsque la norme de τ est fixée, la dissipation d'énergie est maximisée si τ et $\dot{\gamma} = 2D(v)$ sont alignés, i.e. si $\tau/|\tau| = \dot{\gamma}/|\dot{\gamma}|$.

À l'échelle macroscopique, la loi de Coulomb énonce que la contrainte $|R_T| = \mu_d |R_N|$ doit être remplacée par une inégalité en cas de non glissement ($|R_T| \leq \mu_s |R_N|$). À l'échelle mésoscopique, lorsqu'il n'y a pas de déformation ($\dot{\gamma} = 0$), la contrainte $|\tau| = \mu p$ est relaxée en une inégalité : $|\tau| \leq \mu p$. Cette caractérisation de la plasticité est classique en mécanique des sols et est nommée critère de Drucker-Prager (DRUCKER et al. 1952).

Il reste alors à déterminer la dépendance du coefficient de friction μ à partir des paramètres de l'écoulement. Pour la loi de friction de Pouliquen, le coefficient $\mu(h, U)$ dépend d'un nombre sans dimension I_g qui peut s'écrire comme le rapport suivant

$$I_g = \frac{L|U|}{h\sqrt{gh}}, \quad (0.188)$$

avec L une longueur typique, h la hauteur de l'écoulement et U la vitesse moyenne.

Ces paramètres ne peuvent pas être utilisés ici, car ils sont définis à l'échelle de l'écoulement moyenné sur la profondeur. Or on se place ici à l'échelle mésoscopique, à laquelle les grandeurs précédentes sont des quantités non locales. L'idée est alors de remplacer ces paramètres par une version locale, en utilisant les correspondances suivantes :

- La quantité $\sqrt{gh}$ est la racine carrée de la pression hydrostatique exercée sur le plan incliné, divisée par la densité de l'écoulement. Une version locale est donc $\sqrt{p/\rho_p}$, avec p la pression et ρ_p la densité des grains.
- Une version locale du rapport U/h , qui peut être vu comme un taux d'accroissement de la vitesse, est le taux de cisaillement $\partial u/\partial z$. Pour un écoulement multidimensionnel, le taux de cisaillement est décrit par le tenseur $\dot{\gamma}$. La quantité $|U|/h$ peut donc être remplacée par $|\dot{\gamma}|$.
- À l'échelle locale, une longueur typique est le diamètre des particules d , qui peut donc être utilisé pour remplacer le paramètre L .

Par conséquent, le coefficient de friction local μ est une fonction du nombre sans dimension I défini par

$$I = \frac{|\dot{\gamma}|d}{\sqrt{p/\rho_p}}. \quad (0.189)$$

I est appelé nombre inertiel et permet de caractériser l'écoulement (voir le livre d'ANDREOTTI et al. 2013). Lorsque $I \rightarrow 0$, l'écoulement est dit quasi-statique. Lorsque $I \geq 1$, l'écoulement est rapide et dilué, et dominé par des collisions. Entre ces deux régimes, le milieu est dense et s'écoule de manière analogue à un fluide. Le nombre sans dimension I peut être interprété comme le rapport entre une échelle de temps microscopique liée au réarrangement entre les grains, et une échelle de temps macroscopique liée au cisaillement (GdR MiDi 2004).

3.2.2. La rhéologie $\mu(I)$

Les équations constitutives de la rhéologie $\mu(I)$ sont donc

$$\left\{ \begin{array}{l} \rho = cst, \quad \text{div}(\boldsymbol{v}) = 0, \\ \partial_t(\rho \boldsymbol{v}) + \text{div}(\rho \boldsymbol{v} \otimes \boldsymbol{v} + p \text{Id})^T = \text{div}(\boldsymbol{\tau})^T + \rho \boldsymbol{g}, \\ \boldsymbol{\tau} = \mu(I) p \frac{\dot{\gamma}}{|\dot{\gamma}|} \quad \text{si } \dot{\gamma} \neq 0, \\ |\boldsymbol{\tau}| \leq \mu(I) p \quad \text{si } \dot{\gamma} = 0, \\ I = \frac{d|\dot{\gamma}|}{\sqrt{p/\rho_p}}. \end{array} \right. \quad (0.190)$$

Pour fermer les équations, il reste à déterminer comment le coefficient de friction μ dépend du nombre inertiel I . Pour cela, on peut mesurer expérimentalement μ et I , et interpoler les données obtenues. Une autre méthode consiste à étudier les solutions stationnaires uniformes de la rhéologie $\mu(I)$, et à prescrire une loi $\mu(I)$ permettant de retrouver les comportements observés à l'échelle des écoulements

moyennés sur la profondeur. Considérons donc une solution stationnaire et uniforme des équations constitutives (0.190), dans le cadre d'un écoulement bidimensionnel sur plan incliné. On reprend les notations de la section 1.5, et on cherche donc une solution qui peut dépendre de la variable z , mais pas des variables t (stationnaire) et x (uniforme). La contrainte $\text{div}(v) = 0$ donne alors $\partial_z w = 0$, ce qui couplé à la condition de non pénétration $w(0) = 0$ impose $w = 0$. Le champ de vitesses est donc de la forme $v = (u(z), 0)^T$. On en déduit que

$$\dot{\gamma} = \begin{pmatrix} 0 & \partial_z u \\ \partial_z u & 0 \end{pmatrix}, \quad \text{et} \quad |\dot{\gamma}| = |\partial_z u|. \quad (0.191)$$

De plus, les termes d'accélération sont nuls et l'équation de la quantité de mouvement est réduite à $\text{div}(pId - \tau)^T = \rho g$. En supposant que la vitesse est non nulle et puisque $u(0) = 0$ (condition de non glissement (0.65)), on obtient $|\dot{\gamma}| = |\partial_z u| \neq 0$. Les contraintes tangentielle et normale sont donc données par

$$\tau_{xz} = \rho g(h - z) \sin \theta \quad \text{et} \quad p = \rho g(h - z) \cos \theta, \quad (0.192)$$

cette dernière correspondant à la pression hydrostatique. Par conséquent, $\mu(I) = |\tau|/p = \tan \theta$. On retrouve donc la condition d'équilibre $\mu = \tan \theta$ dérivée précédemment. Avec la rhéologie $\mu(I)$, cette condition est vérifiée en tout point de l'écoulement. On en déduit que I est constant et donné par $I_\theta = \mu^{-1}(\tan \theta)$. En utilisant la définition de I et la condition $u(0) = 0$, on obtient le profil de vitesse donné par

$$u(z) = \frac{2I_\theta \sqrt{\phi g \cos \theta}}{3d} (h^{3/2} - (h - z)^{3/2}) \quad (0.193)$$

(profil de Bagnold), avec $\phi = \rho/\rho_p$ la fraction volumique solide. Notons que le profil de Bagnold correspond bien aux profils observés expérimentalement ou numériquement (voir par exemple GdR MiDi 2004), ce qui est une validation de la rhéologie dans cette configuration. La vitesse moyenne est alors donnée par

$$U = \frac{1}{h} \int_0^h u(z) dz = \frac{2I_\theta \sqrt{\phi g \cos \theta}}{5d} h^{3/2}. \quad (0.194)$$

Cela confirme la relation (0.181) entre le nombre de Froude et la hauteur obtenue par POULIQUEN 1999b. On en déduit qu'à l'équilibre, le nombre inertiel est donné par

$$I_\theta = \frac{5dU}{2h^{3/2} \sqrt{\phi g \cos \theta}}. \quad (0.195)$$

La fonction $\mu(I)$ peut alors être déterminée en faisant coïncider $\mu(I_\theta)$ avec la loi de friction $\mu(h, U)$ de POULIQUEN 1999b. En prenant par exemple l'expression (0.180) de

h_{stop} , on obtient la formule

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + \frac{I_m}{I}}, \quad \text{avec} \quad I_m = \frac{5\beta d}{2L\sqrt{\phi}}, \quad \text{et donc} \quad I_\theta = I_m \frac{\tan\theta - \mu_1}{\mu_2 - \tan\theta}. \quad (0.196)$$

La stabilité de cette solution pour les équations constitutives de la rhéologie $\mu(I)$ (0.190) a été étudiée par FORTERRE 2006. La rhéologie $\mu(I)$ linéarisée prédit en effet que l'instabilité des ondes de surfaces a lieu lorsque le nombre de Froude $\mathcal{F}$ est supérieur au nombre de Froude critique $\mathcal{F}_c$ donné par

$$\mathcal{F}_c = \frac{4}{5} \sqrt{1 - \frac{3}{2} \tan^2 \theta + \frac{3}{4} \frac{(\tan\theta - \mu_1)(\mu_2 - \tan\theta)}{\mu_2 - \mu_1}}, \quad (0.197)$$

ce qui est en accord avec les données expérimentales de FORTERRE et POULIQUEN 2003. La rhéologie linéarisée prédit également avec une bonne précision le taux de croissance et la vitesse de phase de la perturbation. En particulier, on obtient théoriquement la fréquence de coupure observée expérimentalement, ce qui n'est pas le cas avec le modèle (0.183) de type Saint Venant avec loi de friction présenté précédemment.

3.2.3. Conséquence pour les modèles moyennés

On voit avec le paragraphe précédent que, dans la configuration du plan incliné, la rhéologie $\mu(I)$ fait de meilleures prédictions que l'approche moyennée sur la profondeur. En effet, en plus de décrire l'état du milieu à l'intérieur de l'écoulement (profil de vitesse, champs de pression et cisaillement), elle permet de prévoir avec précision la réponse linéaire à une perturbation (Froude critique, fréquence de coupure). Cependant, comme expliqué en début d'introduction, la résolution numérique des équations constitutives de la rhéologie $\mu(I)$ est coûteuse en temps de calcul. Un autre défaut de la rhéologie $\mu(I)$ est son caractère mal posé. En effet, BARKER, SCHAEFFER, BOHÓRQUEZ et al. 2015 montrent que la rhéologie $\mu(I)$ admet l'instabilité de Hadamard : le taux de croissance linéaire des perturbations tend vers l'infini lorsque la fréquence de la perturbation tend vers l'infini. Cette instabilité a par exemple été observée numériquement par N. MARTIN et al. 2017, qui montrent que des oscillations du champ de vitesse et du cisaillement apparaissent lorsque la taille de maille utilisée est réduite (voir Figure 8). Les auteurs précisent cependant que ces oscillations ont lieu lorsque la taille d'une maille est de l'ordre de quelques diamètres de grains, et qu'à cette échelle la description continue n'est pas la plus pertinente.

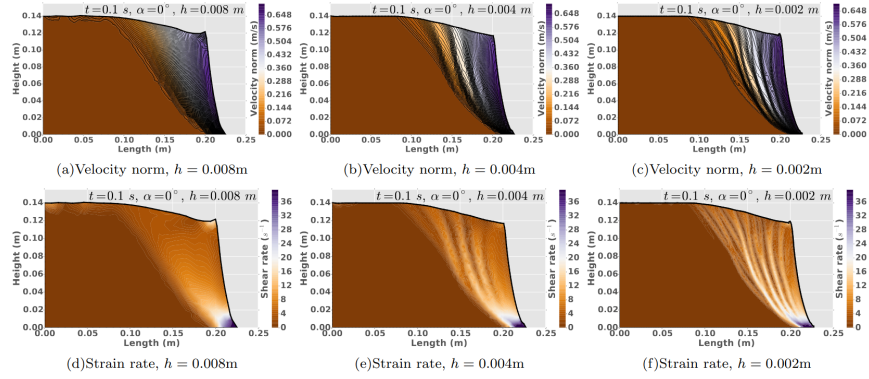


FIGURE 8. – Instabilité d’Hadamard pour la rhéologie $\mu(I)$ dans la simulation de l’effondrement d’une colonne de grains. h est ici la taille des mailles. Figure tirée de N. MARTIN et al. 2017

Au contraire des équations de la rhéologie $\mu(I)$, le modèle de type Saint Venant avec friction (0.183) est bien posé (structure hyperbolique, voir théorème 1) et rapide à résoudre, mais moins précis (pas de fréquence de coupure). Afin de combiner le meilleur des deux approches, une solution est d’utiliser les prédictions de la rhéologie $\mu(I)$ pour affiner les modèles de type Saint Venant. Citons deux exemples d’amélioration :

- Grâce à la rhéologie $\mu(I)$, on connaît le profil de vitesse au sein d’un écoulement stationnaire uniforme (profil de Bagnold, voir équation (0.193)). Pour un tel profil, le facteur de forme α peut être calculé et vaut $5/4$. Dans l’article de SAINGIER et al. 2016, cette valeur de α est utilisée afin de prédire la forme d’un front sec, en bon accord avec des mesures expérimentales. La prédiction est meilleure qu’avec $\alpha = 1$, car utiliser $\alpha = 5/4$ permet de prendre en compte l’aplatissement du profil avec l’augmentation du nombre de Froude. Toutefois, le front sec n’est bien défini que lorsque $\alpha = 1$. Pour $\alpha \neq 1$, les auteurs montrent que le modèle prévoit l’apparition d’un film précurseur de longueur infinie, qui ne correspond pas aux observations expérimentales. À partir de simulations de fronts secs basées sur une approximation des équations de la rhéologie $\mu(I)$ (modèle granulaire RNS/P), LAGRÉE, SAINGIER et al. 2017 montrent que le coefficient α vaut bien $5/4$, sauf près du front où il tend vers 1, et il n’y a alors aucun film précurseur. Notons que dans le cadre des équations avec tenseur de Reynolds, $\alpha = 5/4$ correspond à $P = U^2/4$ et $\psi = U^2/4h^2$ (et $\alpha = 1$ correspond à $\psi = 0$).
- Afin de modéliser la fréquence de coupure, on peut ajouter au modèle de type Saint Venant (0.183) un terme de diffusion horizontale permettant d’amortir les hautes fréquences. Ce terme peut être calculé en faisant une approximation de l’intégrale de τ_{xx} , qui est donné par la rhéologie. Cette approche a été utilisée par FORTERRÉ 2006, qui a proposé un terme diffusif D donné par la formule

$$D = a\rho_p d^2 \frac{\tan\theta}{I_\theta^2} \partial_x \left(\frac{U}{h} \partial_x (hU) \right), \quad (0.198)$$

où a est un paramètre sans dimension non spécifié. Le modèle correspondant permet de retrouver la fréquence de coupure avec une bonne précision lorsque $a = 0.1$. GRAY et EDWARDS 2014 proposent un terme diffusif donné par

$$D = \partial_x \left(\frac{5d \sin \theta}{9I_\theta} \sqrt{\frac{g}{\phi \cos \theta}} h^{3/2} \partial_x U \right), \quad (0.199)$$

qui est obtenu de manière consistante avec l'intégrale de τ_{xx} pour la rhéologie $\mu(I)$ et qui prédit bien la fréquence de coupure, sans paramètre d'ajustement. Notons toutefois que, paradoxalement, les prédictions du nombre de Froude critique et de la fréquence de coupure de ces modèles est beaucoup moins bonne lorsqu'un facteur de forme $\alpha = 5/4$ est utilisé (par exemple $F_c = 2/\sqrt{5} \approx 0.89$).

Les deux exemples précédents montrent que la prise en compte de la rhéologie permet d'améliorer les modèles d'équations moyennées sur la profondeur, et que l'utilisation d'un facteur de forme $\alpha = 5/4$ n'est pas toujours pertinente. Un résultat de cette thèse, présenté dans le chapitre 3, est la dérivation d'un modèle à trois équations prenant en compte le profil de Bagnold via l'approche du tenseur de Reynolds présentée dans la section 2.3. Le modèle est obtenu à partir d'un développement asymptotique des équations de la rhéologie $\mu(I)$ en puissances du paramètre de couche mince ε introduit dans le paragraphe 1.5. On a vu dans ce paragraphe que, dans les variables adimensionnées, l'équation horizontale de la quantité de mouvement s'écrit (0.70)

$$\varepsilon F^2 [\partial_{\tilde{t}} \tilde{u} + \partial_{\tilde{x}}(\tilde{u}^2) + \partial_{\tilde{z}}(\tilde{u} \tilde{w})] = \sin \theta + \varepsilon \partial_{\tilde{x}} \tilde{\sigma}_{xx} + \partial_{\tilde{z}} \tilde{\sigma}_{xz}, \quad (0.200)$$

ce qui donne après intégration sur la profondeur (voir section 1.5)

$$\varepsilon F^2 [\partial_{\tilde{t}}(\tilde{h} \tilde{U}) + \partial_{\tilde{x}}(\tilde{h} \tilde{U}^2 + \tilde{h}^3 \tilde{\psi})] = \tilde{h} \sin \theta - \tilde{\sigma}_{xz}(0) + \varepsilon \partial_{\tilde{x}} \left(\int_0^{\tilde{h}} \tilde{\sigma}_{xx} d\tilde{z} \right). \quad (0.201)$$

Ainsi, on voit que si le terme $\tilde{h} \sin \theta - \tilde{\sigma}_{xz}(0)$ (compétition entre le poids et la friction au fond) n'est approché qu'à l'ordre 0 en ε , des termes d'erreurs d'ordre ε peuvent apparaître. Pour obtenir un modèle avec une bonne précision, il paraît important de ne pas négliger ces termes, qui sont du même ordre de grandeur que les termes d'accélération horizontale. À partir du paramètre ε , on peut définir la consistance d'un modèle moyenné comme suit :

Définition 7. *On dit qu'un modèle d'équations moyennées sur la profondeur est consistant avec la rhéologie $\mu(I)$ jusqu'à l'ordre ε^n si toute solution des équations constitutives de la rhéologie $\mu(I)$ donne, après moyennisation, une solution approchée du modèle moyenné, avec des termes d'erreur d'ordre au plus ε^{n+1} .*

Une conséquence du développement asymptotique de la rhéologie, effectué dans cette thèse jusqu'à l'ordre 1, est le résultat suivant :

RÉSULTAT 4 (DELEAGE ET RICHARD 2025)

Dans l'asymptotique des écoulements en couche mince, l'approximation de la friction au fond $\mu(I)|_{z=0} = \mu(h, U)$, avec $\mu(h, U)$ la loi de friction (0.182), est valide à l'ordre 0 en ε , mais pas à l'ordre 1 en ε . Par conséquent, le modèle de Saint Venant avec loi de friction (0.183), et les versions avec un terme de diffusion donné par (0.198) ou (0.199) ne sont pas consistants à l'ordre 1 avec la rhéologie $\mu(I)$.

L'intérêt d'écrire un modèle consistant à l'ordre 1 peut être justifié par le fait que les termes inertiels sont d'ordre 1. Le second résultat obtenu dans ce cadre est le suivant :

RÉSULTAT 5 (DELÉAGE ET RICHARD, 2025 DELEAGE ET RICHARD 2025)

Il est possible d'écrire un modèle moyenné avec la structure hyperbolique des équations de Reynolds (0.130), la loi de friction $\mu(h, U)$ (0.182) et des termes de correction garantissant la consistance à l'ordre 1 avec la rhéologie $\mu(I)$. Pour ce modèle, décrit dans la partie 2, le seuil d'instabilité des ondes de surface est le même que pour la rhéologie $\mu(I)$ (équation (0.197)).

Les détails de la dérivation et de l'analyse du modèle sont donnés dans le chapitre 3. En particulier, une version avec un terme de diffusion similaire à celui obtenu par GRAY et EDWARDS 2014 et permettant de prédire la fréquence de coupure est présentée.

Remarque 27. *La prise en compte de diffusion dans les modèles de couche mince est courante (voir par exemple BRESCH 2009). Un exemple simple est le modèle de Saint-Venant visqueux, qui s'écrit en une dimension*

$$\begin{cases} \partial_t h + \partial_x(hU) = 0, \\ \partial_t(hU) + \partial_x\left(hU^2 + \frac{gh^2}{2}\right) = \nu \partial_x(h \partial_x U), \end{cases} \quad (0.202)$$

où $\nu > 0$ désigne la viscosité, supposée ici constante. Le terme de diffusion permet d'introduire la vitesse effective, définie ici comme $U + \nu \partial_x \ln(h)$ et proposée par BRESCH et DESJARDINS 2004. Cette vitesse effective satisfait une estimation d'énergie supplémentaire qui permet d'obtenir l'existence globale de solutions faibles pour le modèle (0.202) (voir BRESCH et DESJARDINS 2003). La notion de vitesse effective sera reprise dans la section 3.4 et au chapitre 6.

Le modèle présenté dans le résultat 5 est obtenu grâce à un développement asymptotique effectué au voisinage d'un écoulement stationnaire et uniforme. Pour des configurations très éloignées des écoulements stationnaires et uniformes, certains termes du modèles peuvent diverger et doivent donc être régularisés, afin de garantir un comportement cohérent. Un troisième résultat obtenu dans ce contexte est le suivant :

RÉSULTAT 6

On peut écrire une version régularisée du modèle présenté dans le résultat 5. Cette version du modèle permet de prédire quantitativement les profils de fronts secs, les arrêts d'avalanche et les écoulements accélérés, tout en restant consistante à l'ordre 1 avec la rhéologie $\mu(I)$.

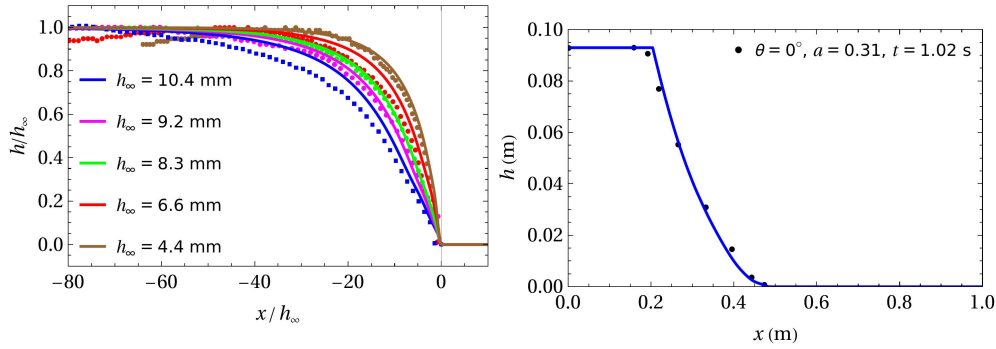


FIGURE 9. – Comparaisons entre les prédictions du modèle régularisé (en traits pleins) et des données expérimentales (points). À gauche, profils de front sec (données de SAINGIER et al. 2016). À droite, arrêt d'avalanche (données de FARIN et al. 2014).

Deux exemples de prédiction du modèle régularisé (profil de front sec et effondrement de colonne) sont présentés dans la figure 9. Ces prédictions sont obtenues par la résolution numérique du modèle. Ici, il est important d'utiliser un schéma numérique pouvant capturer à la fois les fronts secs et l'arrêt de l'écoulement. L'écriture du modèle régularisé, sa résolution numérique et les comparaisons avec les données expérimentales sont présentées en détail au chapitre 4.

3.3. Généralisations compressibles de la rhéologie $\mu(I)$

Dans ce paragraphe, on discute certaines limites de la rhéologie $\mu(I)$ incompressible qui suggèrent qu'utiliser une rhéologie compressible pourrait se révéler plus avantageux dans certaines situations. On présente ensuite deux généralisations compressibles de la rhéologie $\mu(I)$.

3.3.1. Nécessité d'une rhéologie compressible

Si la rhéologie $\mu(I)$ telle que présentée dans le paragraphe précédent permet une description fine des écoulements granulaires dans de nombreux contextes, elle paraît toutefois limitée par certains aspects. Tout d'abord, comme expliqué au paragraphe 3.2.3, cette rhéologie est mal posée au sens de Hadamard. Dans le cas d'un écoulement sur un plan incliné, on a vu que ce problème peut être contourné en utilisant une approximation bien posée de la rhéologie à l'échelle des écoulements moyennés sur

la profondeur. Cependant, il paraît tout de même important de pouvoir écrire un système d'équations constitutives qui soit bien posé dès le départ.

Une autre limite de la rhéologie telle que présentée dans le paragraphe précédent est son caractère incompressible, restreint aux écoulements de densité constante. Si la densité du matériau ρ_p est forcément constante pour des particules solides, la densité de l'écoulement $\rho = \rho_p \phi$ peut en pratique varier. Ici, ϕ désigne la fraction volumique solide, comme définie au paragraphe 2.4 avec la notation α_i . Dans le contexte des écoulements granulaires, c'est la notation ϕ qui est la plus courante, et c'est donc celle qui est utilisée dans cette section.

Dans l'article DA CRUZ et al. 2005, da Cruz et al. montrent à partir de simulations discrètes que, dans le cas d'un cisaillement plan, la fraction volumique solide est reliée au nombre inertiel. Dans le régime étudié, cette relation peut être mise sous la forme d'une fonction affine décroissante :

$$\phi = \phi_c - I \Delta \phi, \quad (0.203)$$

avec $\phi_c > 0$ la fraction volumique maximale et $\Delta \phi > 0$ une constante. La décroissance de ϕ avec I peut être interprétée physiquement comme un effet de dilatance : si le cisaillement (et donc le nombre inertiel) augmente, la fraction de volume solide diminue. L'expansion du milieu facilite alors le mouvement relatif des grains. DA CRUZ et al. 2005 proposent donc une loi constitutive, appelée ici la rhéologie $\mu(I) - \phi(I)$ et donnée par les relations

$$|\tau| = \mu(I) p \quad \text{et} \quad \phi = \phi(I), \quad \text{avec} \quad I = \frac{d|\dot{\gamma}|}{\sqrt{p/\rho_p}}. \quad (0.204)$$

Notons que cette loi constitutive concerne uniquement des quantités scalaires. Son utilisation est donc limitée à la description d'écoulements unidimensionnels. Le terme unidimensionnel est ici à prendre au sens large, et désigne un écoulement pour lequel il existe un système de coordonnées telles que le vecteur vitesse ne possède qu'une composante non nulle et ne dépend que d'une coordonnée spatiale. Ce type de configuration est aussi appelée un cisaillement simple, parce que le tenseur des taux de déformations $\dot{\gamma}$ est réduit dans ce cas à un unique scalaire. C'est le cas par exemple pour les configurations étudiées dans l'article du GdR MIDi 2004, qui en plus de l'écoulement sur plan incliné comprennent le cisaillement plan, le cisaillement annulaire, la chute verticale, l'écoulement sur une pile de grains et le tambour rotatif.

Dans le cas d'un écoulement sur plan incliné, la rhéologie $\mu(I) - \phi(I)$ prédit que le nombre inertiel $I = I_\theta$ est constant, et tel que $\mu(I_\theta) = \tan \theta$. Par conséquent, c'est également le cas de la fraction volumique $\phi = \phi(I_\theta)$. Cela donne donc une justification de l'usage de la rhéologie incompressible dans ce contexte. Notons cependant que la valeur de la fraction volumique $\phi(I_\theta)$ dépend de la pente considérée. L'avantage de l'approche incompressible est qu'elle permet de définir la pression sans ambiguïté comme le multiplicateur de Lagrange associé à la contrainte $\text{div}(\nu) = 0$. (voir section 2).

Dans le cas compressible, cette définition de la pression ne peut pas être utilisée, et on doit préciser la relation entre la pression et les autres variables du système. Notons que la pression joue un rôle crucial dans la rhéologie $\mu(I) - \phi(I)$ (0.204), puisqu'elle est utilisée à la fois dans la définition du nombre inertiel I et dans la relation donnant la norme de τ . Il est donc important de bien définir la pression. Expérimentalement, la pression est obtenue en mesurant la contrainte normale. Cependant, dans le cas d'un écoulement multi-dimensionnel, les contraintes normales ne sont pas forcément isotropes (voir par exemple WEINHART et al. 2013), ce qui constitue une difficulté dans l'écriture d'une généralisation tensorielle et compressible des lois constitutives (0.204).

3.3.2. Deux exemples de rhéologie granulaire compressible

On présente ici deux généralisations compressibles de la rhéologie $\mu(I)$. En plus de prendre en compte des variations de fraction de volume (dilatance), on va voir que ces rhéologies sont bien posées (au sens de Hadamard). Pour écrire une rhéologie compressible multi-dimensionnelle de type $\mu(I)$, la première étape est de séparer les déformations isotropes, liées au changement de volume, et les déformations de cisaillement pur (sans changement de volume). En effet, comme expliqué précédemment, la rhéologie $\mu(I) - \phi(I)$ s'applique à des configurations de cisaillement simple, pour lesquelles le volume est constant. On décompose donc

$$\dot{\gamma} = \frac{2\text{div}(v)}{n}\text{Id} + S, \quad \text{i.e.} \quad S = \dot{\gamma} - \frac{2\text{div}(v)}{n}\text{Id}. \quad (0.205)$$

La lettre n désigne ici la dimension d'espace, puisque la notation d est utilisée dans cette section pour le diamètre des particules solides. La relation d'orthogonalité $\text{Tr}(S) = 0$ signifie ici que S décrit un cisaillement à volume constant. Dans le cas compressible multi-dimensionnel, le nombre inertiel peut donc être défini par

$$I = \frac{d|S|}{\sqrt{p^I \rho_p}}. \quad (0.206)$$

Cette définition de I dans le cas compressible a été confirmée par des simulations discrètes (CORTET et al. 2009, LACAZE et al. 2009).

Une première approche, utilisée par NOTT 2009, TRULSSON et al. 2013, HEYMAN et al. 2017, est alors d'utiliser la relation $\phi = \phi(I)$ couplée à la définition de I (0.206) afin d'obtenir une définition de la pression à l'équilibre p_{eq} :

$$p_{eq}(\phi, |S|) = \frac{\rho_p d^2 |S|^2}{I(\phi)^2}, \quad (0.207)$$

avec $\phi \mapsto I(\phi)$ la fonction réciproque de $I \mapsto \phi(I)$. Le tenseur des contraintes est alors

exprimé par la relation suivante :

$$\frac{\sigma}{p_{eq}(\phi, |S|)} = -\text{Id} + \mu(\phi) \frac{S}{|S|} + 2\mu_b(\phi) \frac{\text{div}(v)}{|S|} \text{Id}. \quad (0.208)$$

On voit donc apparaître un coefficient de friction $\mu(\phi)$ lié au cisaillement pur, et un coefficient de friction $\mu_b(\phi)$ lié au changement de volume. Notons qu'ici la dépendance en ϕ est équivalente à une dépendance en I via la relation $\phi = \phi(I)$. Comme pour le cas incompressible, le déviateur τ est aligné avec le tenseur S . La pression vaut ici

$$p = -\frac{1}{n} \text{Tr}(\sigma) = p_{eq}(\phi, |S|) \left(1 - 2\mu_b(\phi) \frac{\text{div}(v)}{|S|} \right). \quad (0.209)$$

Lorsque $\text{div}(v) \neq 0$, la pression p diffère donc de la pression d'équilibre $p_{eq}(\phi, |S|)$. En particulier, pour $\mu_b(\phi) \neq 0$, la relation (0.206) n'est pas vérifiée lorsque $\text{div}(v) \neq 0$ (elle est vérifiée en remplaçant p par p_{eq}). La densité de dissipation d'énergie par unité de temps liée au frottement peut se calculer et vaut ici (voir l'équation (0.45)) :

$$-\sigma : D(v) = p_{eq}(\phi, |S|) \left[\text{div}(v) - \mu(\phi)|S| - 2\mu_b(\phi) \frac{[\text{div}(v)]^2}{|S|} \right] \quad (0.210)$$

Par conséquent, le système est toujours dissipatif ($\sigma : D(v) > 0$, quelles que soient les valeurs de $|S|$ et de $\text{div}(v)$) si et seulement si $\mu(\phi)\mu_b(\phi) > 1/8$, comme remarqué par GODDARD et al. 2018. Le caractère bien posé pour ces équations constitutives (i.e. la non divergence des taux de croissance linéaires dans la limite des ondes courtes) est caractérisé par la proposition suivante (HEYMAN et al. 2017).

Proposition 8. *Les équations constitutives données par les relations (0.207) et (0.208) sont bien posées au sens de Hadamard si, et seulement si, on a la relation*

$$\mu_b(\phi) > 1 - \frac{7}{6}\mu(\phi). \quad (0.211)$$

D'autres relations constitutives ont été proposées par BARKER, SCHAEFFER, SHEARER et al. 2017 (voir aussi ANDRADE et al. 2012), sous le nom de compressible I -dependent rheology (CIDR). Contrairement à la rhéologie étudiée par HEYMAN et al. 2017, l'idée de BARKER, SCHAEFFER, SHEARER et al. 2017, inspirée de la mécanique des sols, est de garder la relation (0.206) définissant I , et de s'autoriser à relaxer la relation $\phi = \phi(I)$ lorsque $\text{div}(v) \neq 0$. Cette relation est donc remplacée par une équation qui donne une définition implicite de la pression en fonction des variables $|S|$, ϕ et $\text{div}(v)$:

$$\text{div}(v) = |S|f(p, \phi, I), \quad (0.212)$$

avec f une fonction à spécifier. Le tenseur des contraintes est alors donné par

$$\sigma = -p\text{Id} + Y(p, \phi, I) \frac{S}{|S|}, \quad (0.213)$$

avec Y une autre fonction à spécifier. BARKER, SCHAEFFER, SHEARER et al. 2017 donnent alors les critères suivants afin d'obtenir un système bien posé :

Proposition 9. *Supposons que les fonctions f et Y vérifient les conditions suivantes :*

$$\partial_p Y - \frac{I}{2p} \partial_I Y = f + I \partial_I f, \quad (0.214)$$

$$\partial_I Y > 0, \quad (0.215)$$

$$\partial_p f - \frac{I}{2p} \partial_I f < 0. \quad (0.216)$$

Alors la rhéologie CIDR est bien posée au sens de Hadamard.

Notons que l'équation (0.214) permet de déterminer f à partir de Y , et vice-versa. Pour compléter la rhéologie, il suffit donc par exemple de spécifier la fonction Y donnant la norme du déviateur. Plusieurs expressions pour la fonction Y donnant un modèle bien posé sont proposées dans la littérature (voir par exemple BARKER, SCHAEFFER, SHEARER et al. 2017, SCHAEFFER et al. 2019, et CHUPIN et al. 2024 dans un modèle multiphasique prenant également en compte le gaz interstitiel).

Ces deux exemples montrent qu'il est possible d'écrire une version compressible de la rhéologie $\mu(I)$ qui prend en compte le phénomène de dilatance et qui est bien posée (au sens de Hadamard). On voit également que cette écriture n'est pas unique, et que des considérations supplémentaires sont nécessaires afin de choisir un modèle précis (choix du coefficient $\mu_b(\phi)$ pour la rhéologie de HEYMAN et al. 2017, choix de la fonction $Y(p, \phi, I)$ pour la rhéologie CIDR). Notons que HEYMAN et al. 2017 montrent que les différentes formulations sont équivalentes dans la limite des petits changements de volume.

Une contribution de cette thèse est le résultat suivant, qui constitue une extension du résultat 5 dans le cas d'une rhéologie compressible :

RÉSULTAT 7

Il est possible de modifier le modèle présenté dans le résultat 5 pour obtenir un système d'équations consistantes à l'ordre 1 avec la rhéologie compressible de HEYMAN et al. 2017. Les modifications consistent à

- Remplacer la moyenne sur la profondeur par une moyenne de Favre (i.e. pondérée par la densité du milieu, qui est ici inhomogène) ;
- Redéfinir une des constantes du modèle afin de tenir compte de la dilatance ;
- Ajouter un facteur correctif à la loi de friction.

En particulier, on obtient que le nombre de Froude critique pour l'instabilité des ondes de surface vaut pour cette rhéologie compressible et pour le modèle

moyenné :

$$\mathcal{F}_c = \frac{4}{5} \sqrt{1 - \frac{3}{2} \tan^2 \theta + \frac{3}{4} \frac{(\mu_2 - \tan \theta)(\tan \theta - \mu_1) \tan \theta}{\mu_2 - \mu_1 - (\mu_2 - \tan \theta)(\tan \theta - \mu_1) \partial_\mu \ln \phi|_{\mu=\tan \theta}}} \quad (0.217)$$

Ce modèle est présenté au chapitre 5.

3.4. Rhéologie des suspensions

Les travaux de recherche effectués pour écrire des rhéologies granulaires ont également permis de faire des progrès dans la compréhension du comportement des suspensions granulaires. Une suspension granulaire est un milieu granulaire immergé dans un liquide. Du fait de ce liquide interstitiel, dont la densité est en général bien plus importante que celle de l'air, les propriétés rhéologiques des suspensions granulaires diffèrent de celles des milieux granulaires secs. Néanmoins, les rhéologies de ces deux milieux peuvent être énoncées de manière similaire. Dans ce paragraphe, on présente des équations constitutives décrivant les suspensions granulaires, en mettant en évidence la similarité avec les rhéologies présentées précédemment. On énonce ensuite un modèle jouet de suspension granulaire, qui a été obtenu mathématiquement par homogénéisation. Un des résultats de cette thèse est l'analyse mathématique de ce modèle. On montre en effet l'existence de solutions particulières (ondes progressives), et la stabilité asymptotique de ces solutions dans un espace d'énergie adapté.

3.4.1. Rhéologie $\mu(J)$

Dans le cas d'un écoulement granulaire sec, on a vu précédemment que le milieu est caractérisé par un nombre sans dimension I , appelé nombre inertiel. Ce nombre inertiel permet de déterminer le coefficient de friction $\mu = \mu(I)$, ainsi que la fraction de volume solide $\phi = \phi(I)$. Par analogie avec le nombre inertiel, CASSAR et al. 2005 introduisent dans le cas d'une suspension le nombre visqueux J , défini par

$$J = \frac{\eta_f |\dot{\gamma}|}{p}, \quad (0.218)$$

avec η_f la viscosité du liquide interstitiel, $|\dot{\gamma}|$ le taux de déformation et p la pression. À partir de mesures expérimentales dans le cas d'un cisaillement simple, François BOYER et al. 2011 montrent pour les suspensions étudiées les relations suivantes :

$$|\tau| = \mu(J)p, \quad \text{et} \quad \phi = \phi(J). \quad (0.219)$$

On voit que ces équations constitutives sont les mêmes que pour un milieu granulaire sec (0.204), en remplaçant le nombre inertiel I par le nombre visqueux J . Boyer, Guazzelli et Pouliquen (François BOYER et al. 2011) proposent pour le coefficient de

friction la loi

$$\mu(J) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + I_m/J} + J + \frac{5}{2}\phi_c\sqrt{J}. \quad (0.220)$$

Comparativement à (0.196), les deux premiers termes, dépendant des coefficients μ_1 et μ_2 , modélisent une friction provenant du contact entre les particules et sont écrits de manière similaire à la loi de friction pour un milieu granulaire sec, tandis que les termes restants modélisent des interactions hydrodynamiques. La constante ϕ_c désigne ici la fraction volumique solide maximale, obtenue pour $J = 0$. En effet, la loi proposée par François BOYER et al. 2011 pour la fraction volumique solide s'écrit :

$$\phi = \phi(J) = \frac{\phi_c}{1 + \sqrt{J}}. \quad (0.221)$$

Ces relations sont présentées dans les variables $p, |\dot{\gamma}|$, qui correspondent à des mesures faites en contrôlant la pression de confinement. Dans le cas des suspensions, il est également courant d'utiliser les variables $\phi, |\dot{\gamma}|$, qui correspondent à une fraction volumique solide imposée. Dans ce cas, la relation (0.221) donnant ϕ en fonction J permet d'écrire

$$J = J(\phi) = \left(\frac{\phi_c - \phi}{\phi} \right)^2. \quad (0.222)$$

En utilisant la définition du nombre J (0.218), on en déduit que la pression s'écrit

$$p = \frac{\eta_f |\dot{\gamma}|}{J(\phi)} = \eta_f |\dot{\gamma}| \left(\frac{\phi}{\phi_c - \phi} \right)^2. \quad (0.223)$$

De même, la norme du déviateur est donnée par

$$|\tau| = \mu(J)p = \eta_f |\dot{\gamma}| \left[1 + \frac{5}{2}\phi \left(\frac{\phi_c}{\phi_c - \phi} \right) + \mu_c(\phi) \left(\frac{\phi}{\phi_c - \phi} \right)^2 \right], \quad (0.224)$$

avec $\mu_c(\phi) = \mu_1 + (\mu_2 - \mu_1)/[1 + I_m\phi^2(\phi_c - \phi)^{-2}]$. Lorsque ϕ tend vers 0, on voit que la viscosité relative effective $|\tau|/(\eta_f |\dot{\gamma}|)$ est équivalente à $1 + 5\phi/2$. On retrouve la formule théoriquement prédite par EINSTEIN 1905 ; EINSTEIN 1911. Notons qu'une justification rigoureuse de cette formule a été obtenue mathématiquement en considérant N particules en suspension dans un fluide de Stokes, et en utilisant des techniques d'homogénéisation (SÁNCHEZ-PALENCIA 1985 ; LÉVY et al. 1985 ; HILLAIRET et al. 2020) ou un principe variationnel (HAINES et al. 2012). La formule d'Einstein est alors obtenue en calculant la limite de la viscosité du mélange lorsque N tend vers l'infini, tout en gardant une fraction volumique solide ϕ constante. Les termes d'ordres plus élevés en ϕ dans le développement limité de la viscosité du mélange ont également été déterminés (voir par exemple BATCHELOR et al. 1972 ; GÉRARD-VARET et al. 2020 ; DUERINCKX et al. 2020).

Dans la limite $\phi \rightarrow \phi_c$, on voit que la viscosité effective et la pression divergent comme $(\phi_c - \phi)^{-2}$. L'interprétation de cette divergence est que plus on s'approche de la fraction volumique maximale ϕ_c , plus le milieu est difficile à déformer. Cette diver-

gence est confirmée quantitativement par de nombreuses mesures expérimentales, comme c'est présenté par exemple dans l'article de GUAZZELLI et al. 2018 (voir Figure 10).

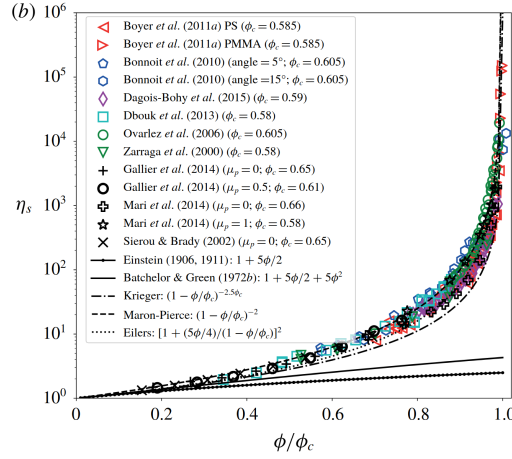


FIGURE 10. – Viscosité relative η_s , donnée par le quotient de la viscosité effective de la suspension par la viscosité du liquide interstitiel, en fonction de la fraction volumique réduite ϕ/ϕ_c . Figure issue de GUAZZELLI et al. 2018

Remarque 28. Dans le cas d'un milieu granulaire sec, la rhéologie $\mu(I) - \phi(I)$ prédit également une divergence de la pression et du déviateur comme $(\phi_c - \phi)^{-2}$ lorsque ϕ tend vers ϕ_c . En effet, en utilisant la définition du nombre inertiel I (0.206) et la relation $\phi = \phi(I)$ (0.203), on peut écrire la pression comme

$$p = \frac{\rho_p d^2 |\dot{\gamma}|^2}{I(\phi)^2}, \quad \text{avec} \quad I(\phi) = \frac{\phi_c - \phi}{\Delta\phi}. \quad (0.225)$$

Par conséquent, la norme du déviateur s'écrit également

$$|\tau| = \frac{\mu(I(\phi))}{I(\phi)^2} \rho_p d^2 |\dot{\gamma}|^2. \quad (0.226)$$

3.4.2. Étude d'un modèle jouet

On présente ici l'étude mathématique d'un modèle jouet possédant quelques aspects non linéaires de la rhéologie $\mu(J)$ présentée au paragraphe précédent. On s'intéresse plus précisément aux effets de divergence de la viscosité effective lorsque la fraction volumique ϕ s'approche de la fraction volumique critique ϕ_c dans un modèle de mécanique des fluides. On ne considère pas ici toutes les équations de la rhéologie $\mu(J)$, dont l'analyse mathématique est pour l'instant un problème ouvert. En particulier, on se restreint au cas unidimensionnel, et on néglige totalement la pression. On considère donc un système d'équations de type Navier-Stokes compressible sans pression, avec un coefficient de diffusion dépendant de la densité et divergent pour

$\phi \rightarrow \phi_c$. Notons que la nature des équations considérées ici est différente de celles présentées dans la section 2.2 (problème hyperbolique), et nécessite par conséquent des outils mathématiques différents. Le système s'écrit

$$\begin{cases} \partial_t \phi + \partial_y(\phi u) = 0, \\ \partial_t(\phi u) + \partial_y(\phi u^2) = \partial_y(\eta(\phi) \partial_y u), \end{cases} \quad (0.227)$$

avec un coefficient de diffusion donné par

$$\eta(\phi) = \frac{f(\phi)}{(1 - \phi)^\gamma}, \quad (0.228)$$

où $f(\phi) > 0$ est une fonction lisse et $\gamma \geq 2$. La fraction volumique maximale ϕ_c est ici prise égale à 1 sans perte de généralité. Notons qu'un modèle similaire a été obtenu par homogénéisation par LEFEBVRE-LEPOT et al. 2008, à partir d'un modèle microscopique de particules solides en suspension dans un fluide de viscosité η_f constante. Le coefficient de diffusion homogénéisé vaut dans ce cas $\eta(\phi) = \eta_f(1 - \phi)^{-1}$.

Pour étudier mathématiquement le modèle (0.227), une première étape est de le reformuler afin de rendre les calculs plus simples. Pour cela, on profite du caractère unidimensionnel pour introduire la coordonnée Lagrangienne de masse x telle que $dx = \rho dy - \rho u dt$, les équations étant posées sur $\mathbb{R}$. Le système peut alors se réécrire

$$\begin{cases} \partial_t w = 0, \\ \partial_t v - \partial_x(\mu(v) \partial_x v) = \partial_x w, \end{cases} \quad (0.229)$$

avec $v = 1/\phi$ le volume massique (ou volume spécifique), $\mu(v) = \phi \eta(\phi)$ le coefficient de diffusion dans les nouvelles coordonnées et $w = u - \mu(v) \partial_x v$ la vitesse effective introduite par BRESCH et DESJARDINS 2004, voir remarque 27. On peut alors chercher des solutions particulières $(\bar{v}, \bar{w})$ de ce système sous la forme d'ondes progressives, i.e. dépendant uniquement de la variable $\xi = x - ct$. $c > 0$ est la vitesse de propagation. On obtient alors que $\bar{w}$ est constant, et que $\bar{v}$ est donné par l'équation différentielle ordinaire (EDO) :

$$\frac{d\bar{v}}{d\xi} = \frac{c(v_+ - \bar{v})}{\mu(\bar{v})}. \quad (0.230)$$

En supposant que $\bar{v}$ atteint une limite finie en $\pm\infty$, on obtient que $\bar{v} \rightarrow v_+ > 1$ en $+\infty$, et $\bar{v} \rightarrow 1$ en $-\infty$. En particulier, ϕ tend en $-\infty$ vers la fraction de volume maximale $\phi_c = 1$. On dit que le profil est partiellement congestionné.

Un résultat de cette thèse est la stabilité asymptotique de ces solutions. La stabilité asymptotique est une propriété importante d'un point de vue mathématique et physique. Elle permet en effet de montrer que des solutions proches des ondes progressives présentées ici peuvent être observées même si la solution exacte est légèrement perturbée. Elle donne également le comportement en temps long de ces solutions. Plus précisément, en considérant une autre solution (v, w) du système (0.227), et en notant $(\delta v, \delta w) = (v, w) - (\bar{v}, \bar{w})$ la perturbation, on montre que si la norme de la perturbation initiale est contrôlée dans un espace adéquat, alors la solu-

tion (v, w) reste proche de l'onde progressive $(\bar{v}, \bar{w})$. La norme utilisée pour contrôler la perturbation initiale permet de borner notamment les quantités suivantes :

- les variables δV et δW , définies comme les intégrales de δv et δw et satisfaisant une estimation d'énergie importante pour le caractère globalement bien-posé, comme identifié par MATSUMURA et al. 1985 ;
- la variable $\delta v / (\bar{v} - 1)$, ce qui permet d'assurer la contrainte $v > 1$ (voir Figure 11).

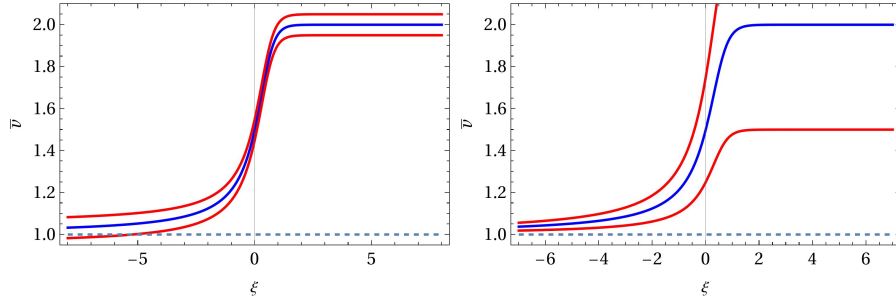


FIGURE 11. – En bleu, profil de l'onde progressive $\bar{v}$. En rouge, intervalle dans lequel peut se trouver la solution v si on suppose une borne du type $\|\delta v\|_{L^\infty(\mathbb{R})} < \delta$ (à gauche, on voit que la contrainte $v > 1$ n'est pas assurée) ou $\|\delta v / (\bar{v} - 1)\|_{L^\infty(\mathbb{R})} < \delta$ (à droite).

Le résultat obtenu peut être énoncé ainsi :

RÉSULTAT 8 (DELEAGE ET MEHMOOD 2024)

Soit $T > 0$ et E_T l'espace d'énergie défini par

$$E_T := \{f \in C([0, T], L^2(\mathbb{R})), \mu(\bar{v})^{1/2} \partial_x f \in L^2((0, T) \times \mathbb{R})\}. \quad (0.231)$$

Il existe $\delta > 0$ tel que, pour toute perturbation initiale $(\delta v_0, \delta w_0)$, si

$$\|\delta V_0\|_{L^2(\mathbb{R})} + \|\delta W_0\|_{H^2(\mathbb{R})} + \left\| \frac{\delta v_0}{\bar{v} - 1} \right\|_{H^1(\mathbb{R})} \leq \delta, \quad (0.232)$$

alors la perturbation $(\delta v, \delta w)$ existe sur $(0, T)$ et est unique dans E_T .

Un résultat plus complet est disponible dans la partie III. On montre notamment que si $\delta w_0 = 0$, alors on peut prendre $T = +\infty$ et on a $\|\delta v(t)\|_{L^\infty(\mathbb{R})} \rightarrow 0$ lorsque $t \rightarrow \infty$.

Première partie

Étude de la structure des équations avec tenseur de Reynolds modélisant des fluides parfaits

1. Symétrisation des équations moyennées de Reynolds

Dans ce chapitre, on montre que les équations moyennées de Reynolds pour les fluides compressibles sont symétrisables au sens de Friedrichs, en dimension $d = 2$ et 3, lorsque la pression s'annule. Ce chapitre est tiré de l'article suivant (DELÉAGE 2024) :

Deléage, É. (2024). Well-posedness of reynolds averaged equations for compressible fluids with a vanishing pressure. *Mathematical Methods in the Applied Sciences*, 47(2), 817-824.

1. Introduction and main results

We study the Reynolds averaged equations for compressible fluids, where third-order correlations are neglected. This system can be written in Eulerian coordinates as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, & (1.1a) \\ \partial_t v + (v \cdot \nabla) v + \frac{1}{\rho} (\nabla p + \operatorname{div}(\rho P)^T) = 0, & (1.1b) \\ \partial_t P + (v \cdot \nabla) P + \frac{\partial v}{\partial x} P + P \left(\frac{\partial v}{\partial x} \right)^T = 0. & (1.1c) \end{cases}$$

The variables are the averaged density $\rho > 0$, the averaged velocity $v \in \mathbb{R}^d$ ($d = 2$ or 3), and P is the Reynolds stress tensor, $P \in S_d^{++}(\mathbb{R})$. The function p is the pressure of the fluid and is a function of the density ρ , through an equation of state $p = p(\rho)$. The map $\rho \mapsto p(\rho)$ is supposed to be of class C^1 and non-decreasing. Typical pressure laws are of the form $p(\rho) = a\rho^\gamma$, with $a > 0$ and $\gamma > 0$ two constants.

The tensor P is a classical Reynolds tensor appearing in the Reynolds averaging of turbulent flows for barotropic compressible fluids (see Pope 2001, Wilcox et al. 1998, Mohammadi et al. 1993). It also appears in the description of free surface shear flows, where the averaging operator is the depth averaging (cf. Annexe C, A7 in Richard 2013 for a derivation of the model). In that latter case, the density ρ must be replaced by the water depth, often denoted h . The pressure is then given by $p(h) = gh^2/2$ (cf. Richard 2013 for instance).

System (1.1) is hyperbolic whenever $p'(\rho) \geq 0$ and P is definite positive, as it has been proved in Richard 2013. Equation (1.1a) shows that the density ρ is conserved.

1. Symétrisation des équations moyennées de Reynolds – 1. Introduction and main results

The conservation of momentum ρv also holds: (1.1b) rewrites as

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + pI_d + \rho P)^T = 0. \quad (1.2)$$

One can also deduce from (1.1) the conservation of energy:

$$\partial_t e + \operatorname{div}(e v + p v + \rho P v) = 0, \text{ with } e := \frac{1}{2} \rho |v|^2 + \rho \mathcal{E}(\rho) + \frac{1}{2} \operatorname{Tr}(\rho P). \quad (1.3)$$

The map $\rho \mapsto \rho \mathcal{E}(\rho)$ is called the volumic internal energy, and is linked to the pressure via the relation $p(\rho) = \rho^2 \mathcal{E}'(\rho)$. The term $\rho |v|^2 / 2$ is the volumic kinetic energy of the fluid, and the term $\operatorname{Tr}(\rho P) / 2$ is the energy associated to the tensor P .

In Gavriluk and Gouin 2020, it was shown that system (1.1) admits a variational formulation, as it is often the case in physics when the energy of a system is conserved. Define the Lagrangian density

$$\mathcal{L}(\rho, v, P) := \frac{1}{2} \rho |v|^2 - \rho \mathcal{E}(\rho) - \frac{1}{2} \operatorname{Tr}(\rho P), \quad (1.4)$$

and the corresponding action

$$\mathcal{A} := \int_{t_0}^{t_1} \int_{\mathbb{R}^d} \mathcal{L} \, dx \, dt. \quad (1.5)$$

Then one can show that (1.1b) is the Euler-Lagrange equation given by the stationary action principle applied to the action (1.5), under the two constraints (1.1a) and (1.1c).

The tensor P admits an additional conservation law, sometimes called conservation of enstrophy, that is a consequence of (1.1c) and can be written

$$\partial_t \left(\frac{\det P}{\rho^2} \right) + v \cdot \nabla \left(\frac{\det P}{\rho^2} \right) = 0. \quad (1.6)$$

Note that equation (1.1c) implies that the symmetry of P is conserved by the evolution. Equation (1.6) then implies that, if $P(t) \in S_d^{++}(\mathbb{R})$ for some instant t , then this property is true for all times.

It has been proved in Gavriluk, Ivanova, et al. 2018 that system (1.1) does not admit any further conservation law. Thus, in dimension $d = 2$ or 3 , system (1.1) is not conservative. Hence the usual symmetrisation method of Godunov (cf. Godunov 1961) and Lax and Friedrichs (cf. Friedrichs et al. 1971) for hyperbolic systems of conservation laws (see for instance Serre 1999, pages 83-84) can not be used here.

However, one can show that system (1.1) is Friedrichs-symmetrizable when the pressure vanishes. More precisely, we state the following theorem :

Theorem 1.1. *Let $d = 2$ or 3 . Suppose that ρ takes values in $\mathbb{R}_+^*$ and P takes values in $S_d^{++}(\mathbb{R})$. Then the two following properties are equivalent:*

1. System (1.1) is Friedrichs-symmetrizable

1. Symétrisation des équations moyennées de Reynolds – 1. Introduction and main results

2. *The pressure p is constant: $p'(\rho) = 0$, or the tensor P is a scalar matrix, i.e. there exists $\lambda = \lambda(t, x) \in \mathbb{R}$ such that $P = \lambda I_d$.*

Let us make few comments about this theorem:

- The tensor $P = \lambda I_d$ is a solution of system (1.1) only for trivial velocities; hence, for applications of this result, the case $p' = 0$ seems more interesting.
- This property holds in the variables (ρ, v, P) . It could be possible that system (1.1), written in different variables, appears to be symmetrizable even when $p' \neq 0$.
- When $d = 1$, system (1.1) is symmetrizable, even when $p' \neq 0$. In fact, one can prove that one-dimensional hyperbolic systems are always Friedrichs-symmetrizable (cf. Panov n.d.). In higher dimension $d \geq 2$, this does not hold anymore.

As a consequence of Theorem 1.1, we have the following result regarding the well-posedness of system (1.1):

Proposition 1.1. *Let $d = 2$ or 3 , $s > 1 + d/2$ and $\mathcal{U} := \mathbb{R}_+^* \times \mathbb{R}^d \times S_d^{++}(\mathbb{R})$. Let $\bar{Y} := (\bar{\rho}, \bar{u}, \bar{P}) \in \mathcal{U}$ and Y_0 taking values in $\mathcal{U}$ such that $Y_0 - \bar{Y} \in H^s(\mathbb{R}^d)$. We consider the Cauchy problem associated to (1.1) with initial data Y_0 . There exists $T > 0$ such that (1.1) with $p' = 0$ has a unique classical solution $Y(t)$ in $\mathcal{C}^1([0, T] \times \mathbb{R}^d)$ with values in $\mathcal{U}$ achieving the initial data $Y(0) = Y_0$. Furthermore, $Y - \bar{Y}$ belongs to $\mathcal{C}([0, T], H^s(\mathbb{R}^d)) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}^d))$.*

Proposition 1.1 is a consequence of Theorem 1.1 and of the theory of well-posedness of Kato for quasilinear evolution equations (cf. Kato 2006). For a detailed proof of this result, see for instance Sect. 10 in Benzoni-Gavage et al. 2006.

System (1.1) has been used over the last years in the modeling of turbulent flows, including for numerical simulations (see Richard and Gavriluk 2013, Ivanova, Gavriluk, et al. 2017, Gavriluk, Ivanova, et al. 2018, Ivanova and Gavriluk 2019). However, the well-posedness of (1.1) in dimension $d \geq 2$ is still uncertain today. Proposition 1.1 states an answer to this question in the case of a vanishing pressure. One could also obtain system (1.1) with $p' = 0$ when modeling a fluid for which the pressure gradient ∇p is negligible compared to the turbulence of the fluid $\text{div}(\rho P)$ in (1.1b). This amounts to consider the high Mach number limit. This limit was studied by various authors in the Euler and Navier-Stokes cases and can lead to a vanishing pressure gradient (Humpherys et al. 2010, Qu et al. 2020). The hypothesis $p' = 0$ can also be found in the literature in a model called “pressureless gas dynamics”. This system appears in different physical contexts. It is used in cosmology in order to model galaxies formation in the presence of gravitational instability (see Zel’Dovich 1970). These equations also arise as an hydrodynamic limit of kinetic equations, when considering a cold plasma (see Bouchut 1994), or the granular gases equation (see Jabin et al. 2017). Another field of applications is the study of collective behaviors, in which the pressureless Euler equations appear with additional non-local dissipative terms (Carrillo et al. 2017, Figalli et al. 2018). Eventually, the study of the pressureless equations is motivated by numerical analysis: some numerical solvers for Euler equations also use the case $p' = 0$ as an intermediate step in their splitting method,

1. Symétrisation des équations moyennées de Reynolds – 2. Proof of the theorem

this allows to decouple the convective subsystem from the acoustic one (Baraille et al. 1992, see also Iampietro et al. 2018).

The mathematical analysis of the pressureless Euler equations is constrained by the fact that the system is only weakly hyperbolic. As a consequence, phenomena like creation of vacuum or high concentrations (delta shocks) can occur (see for instance Zel'Dovich 1970, Bouchut 1994, Bouchut, Jin, et al. 2003, Berthon, Breuß, et al. 2006). System (1.1) with $p' = 0$ can thus be seen as a hyperbolic version of the pressureless gas model. In this sense, the Reynolds tensor P brings more regularity to the model.

Notations

We denote $\partial_i := \partial/\partial x_i$ the partial derivative of the variable x_i , for $1 \leq i \leq d$. If $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a scalar function, we denote by $\nabla f \in \mathbb{R}^d$ the gradient of f , i.e. the vector field of components $\partial_i f$, $1 \leq i \leq d$.

If $Z = (Z_1, \dots, Z_n) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a vector field, the divergence of Z is the scalar function defined by

$$\operatorname{div}(Z) := \partial_1 Z_1 + \dots + \partial_n Z_n.$$

We also denote by $\partial Z / \partial x$ the Jacobian matrix of Z , i.e. the matrix of coefficients $(\partial Z / \partial x)_{i,j} = \partial Z_i / \partial x_j$, for $1 \leq i, j \leq d$.

If $Z = (Z_i)_{1 \leq i \leq d}$ and $Z' = (Z'_i)_{1 \leq i \leq d}$ are two vector fields, we denote $Z \otimes Z'$ the second order tensor defined by $Z \otimes Z' := Z(Z')^T$, i.e. the matrix of coefficients $(Z \otimes Z')_{i,j} = Z_i Z'_j$, for $1 \leq i, j \leq d$.

If $A : \mathbb{R}^d \rightarrow M_d(\mathbb{R})$ is a second order tensor, we defined the divergence of A as the line vector of $\mathbb{R}^d$ whose i -th component is given by the divergence of the i -th column of A .

For any positive integer d , we denote $I_d \in M_d(\mathbb{R})$ the identity matrix of size d . We denote $S_d^{++}(\mathbb{R})$ the set of symmetric definite positive matrices, i.e. the symmetric matrices of size d with a positive spectrum.

2. Proof of the theorem

We now give the proof of Theorem 1.1.

Proof. We write system (1.1) in matricial form:

$$\partial_t Y + A(Y, \nabla) Y = 0, \text{ with } Y := \begin{pmatrix} \rho \\ v \\ \tilde{P} \end{pmatrix} \in \mathbb{R}^{1+d+d(d+1)/2}$$

and, if $\xi = (\xi_i)_{1 \leq i \leq d} \in \mathbb{R}^d$,

$$A(Y, \xi) := \begin{pmatrix} v \cdot \xi & \rho \xi^T & 0 \\ \frac{1}{\rho}(p'(\rho)I_d + P)\xi & (v \cdot \xi)I_d & C(\xi) \\ 0 & D(\xi) & (v \cdot \xi)I_{d(d+1)/2} \end{pmatrix}. \quad (1.7)$$

1. Symétrisation des équations moyennées de Reynolds – 2. Proof of the theorem

When $d = 2$, the symmetric matrix $P = (P_{ij})_{1 \leq i, j \leq 2}$ can be identified as a vector $\tilde{P} \in \mathbb{R}^3$ given by

$$\tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{22} \end{pmatrix}.$$

The matrices $C(\xi)$ and $D(\xi)$ are then given by

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & 0 \\ 0 & \xi_1 & \xi_2 \end{pmatrix} \text{ and } D(\xi) := \begin{pmatrix} 2P_{11}\xi_1 + 2P_{12}\xi_2 & 0 \\ P_{21}\xi_1 + P_{22}\xi_2 & P_{11}\xi_1 + P_{12}\xi_2 \\ 0 & 2P_{12}\xi_1 + 2P_{22}\xi_2 \end{pmatrix}.$$

When $d = 3$, the symmetric matrix P can be identified as a vector $\tilde{P} \in \mathbb{R}^6$:

$$\tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{13} \\ P_{22} \\ P_{23} \\ P_{33} \end{pmatrix}.$$

The matrices $C(\xi)$ and $D(\xi)$ are then given by

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ 0 & \xi_1 & 0 & \xi_2 & \xi_3 & 0 \\ 0 & 0 & \xi_1 & 0 & \xi_2 & \xi_3 \end{pmatrix}$$

and

$$D(\xi) := \begin{pmatrix} 2(P\xi)_1 & 0 & 0 \\ (P\xi)_2 & (P\xi)_1 & 0 \\ (P\xi)_3 & 0 & (P\xi)_1 \\ 0 & 2(P\xi)_2 & 0 \\ 0 & (P\xi)_3 & (P\xi)_2 \\ 0 & 0 & 2(P\xi)_3 \end{pmatrix},$$

where $(P\xi)_i$ denotes the i -th component of the vector $P\xi$.

We first show the implication (2) $\Rightarrow$ (1)

Namely, if $p' = 0$ or $P = \lambda I_d$, then system (1.1) is Friedrichs-symmetrizable.

We thus suppose that $p' = 0$ or $P = \lambda I_d$.

Consider $S = S(Y) \in S_n^{++}(\mathbb{R})$ (with $n = (d+1)(d+2)/2$) defined as a block matrix, compatible with A :

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S_2 & 0 \\ 0 & 0 & S_3 \end{pmatrix}, \text{ with } S_2 \in M_d(\mathbb{R}) \text{ and } S_3 \in M_{d(d+1)/2}(\mathbb{R}). \quad (1.8)$$

1. Symétrisation des équations moyennées de Reynolds – 2. Proof of the theorem

Note that S is symmetric definite positive if and only if the matrices S_2 and S_3 are also symmetric definite positive.

We can now compute the product SA by block matrix multiplication. We obtain that

$$SA = \begin{pmatrix} v \cdot \xi & \rho \xi^T & 0 \\ \frac{1}{\rho} S_2(p'(\rho)Id + P)\xi & (v \cdot \xi)S_2 & S_2C(\xi) \\ 0 & S_3D(\xi) & (v \cdot \xi)S_3 \end{pmatrix} \quad (1.9)$$

Case $d = 2$.

Let us choose

$$S_2 := \mu P^{-1} \text{ and } S_3 := \mu \begin{pmatrix} \frac{1}{2} q_{11}^2 & q_{12} q_{11} & \frac{1}{2} q_{12}^2 \\ q_{12} q_{11} & q_{11} q_{22} + q_{12}^2 & q_{12} q_{22} \\ \frac{1}{2} q_{12}^2 & q_{12} q_{22} & \frac{1}{2} q_{22}^2 \end{pmatrix},$$

where we denoted

$$P^{-1} = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} \in S_2^{++}(\mathbb{R}), \quad (1.10)$$

and

$$\mu := \begin{cases} \rho^2 & \text{when } p'(\rho) = 0, \\ \rho^2 \frac{\lambda}{p'(\rho) + \lambda} & \text{when } P = \lambda Id. \end{cases} \quad (1.11)$$

The matrix S_2 is symmetric definite positive.

We see that the principal minors of $\mu^{-1} S_3$ are given by $M_1 = \frac{1}{2} q_{11}^2 > 0$, $M_2 = \frac{1}{2} q_{11}^2 (q_{11} q_{22} - q_{12}^2) > 0$, and $M_3 = \frac{1}{4} (q_{11} q_{22} - q_{12}^2)^3 > 0$ (recall that P is positive definite). Hence by Sylvester's criterion, S_3 is definite positive, and S , defined by (1.8), is a positive definite matrix.

We compute that

$$\frac{1}{\rho} S_2(p'(\rho)Id + P)\xi = \rho \xi, \quad (1.12)$$

$$S_2C(\xi) = \mu \begin{pmatrix} q_{11}\xi_1 & q_{11}\xi_2 + q_{12}\xi_1 & q_{12}\xi_2 \\ q_{12}\xi_1 & q_{12}\xi_2 + q_{22}\xi_1 & q_{22}\xi_2 \end{pmatrix},$$

and, since $q_{i1}P_{1j} + q_{i2}P_{2j} = \delta_{ij}$ by (1.10),

$$S_3D(\xi) = \mu \begin{pmatrix} q_{11}\xi_1 & q_{12}\xi_1 \\ q_{12}\xi_1 + q_{11}\xi_2 & q_{22}\xi_1 + q_{12}\xi_2 \\ q_{12}\xi_2 & q_{22}\xi_2 \end{pmatrix} = [S_2C(\xi)]^T.$$

Hence for any $Y \in \mathcal{U}$, and for any $\xi \in \mathbb{R}^2$, the matrix $S(Y)$ is symmetric definite positive and (1.9) shows that the matrix $S(Y)A(Y, \xi)$ is symmetric. As a consequence, (1.1) is Friedrichs-symmetrizable when $d = 2$.

1. Symétrisation des équations moyennées de Reynolds – 2. Proof of the theorem

Case $d = 3$.

Define again S by equation (1.8). Choose $S_2 = \mu P^{-1}$ with μ as in (1.11), such that S_2 is symmetric definite positive and (1.12) holds again. Define S_3 by

$$S_3 := \mu \begin{pmatrix} \frac{1}{2}q_{11}^2 & q_{11}q_{12} & q_{11}q_{13} & \frac{1}{2}q_{12}^2 & q_{12}q_{13} & \frac{1}{2}q_{13}^2 \\ q_{11}q_{12} & q_{11}q_{22} + q_{12}^2 & q_{11}q_{23} + q_{12}q_{13} & q_{12}q_{22} & q_{12}q_{23} + q_{22}q_{13} & q_{13}q_{23} \\ q_{11}q_{13} & q_{11}q_{23} + q_{13}q_{12} & q_{11}q_{33} + q_{13}^2 & q_{12}q_{23} & q_{12}q_{33} + q_{23}q_{13} & q_{13}q_{33} \\ \frac{1}{2}q_{12}^2 & q_{12}q_{22} & q_{12}q_{23} & \frac{1}{2}q_{22}^2 & q_{22}q_{23} & \frac{1}{2}q_{23}^2 \\ q_{12}q_{13} & q_{12}q_{23} + q_{13}q_{12} & q_{12}q_{33} + q_{13}q_{23} & q_{22}q_{23} & q_{22}q_{33} + q_{23}^2 & q_{23}q_{33} \\ \frac{1}{2}q_{13}^2 & q_{13}q_{23} & q_{13}q_{33} & \frac{1}{2}q_{23}^2 & q_{23}q_{33} & \frac{1}{2}q_{33}^2 \end{pmatrix},$$

where we denoted again

$$P^{-1} := \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{pmatrix}.$$

We see that S_3 is symmetric. Furthermore, the principal minors of $\mu^{-1}S_3$ are given by $M_1 = q_{11}^2/2 > 0$, $M_2 = q_{11}^2(q_{11}q_{22} - q_{12}^2)/2 > 0$, $M_3 = q_{11}^3 \det(P^{-1})/2 > 0$, $M_4 = q_{11}(q_{11}q_{22} - q_{12}^2)^2 \det(P^{-1})/4 > 0$, $M_5 = (q_{11}q_{22} - q_{12}^2)^2 \det(P^{-1})^2/4 > 0$ and finally $M_6 = \det(P^{-1})^4/8 > 0$. Hence, by Sylvester's criterion, S_3 is definite positive and S is symmetric definite positive.

We also check that

$$S_3 D(\xi) = [S_2 C(\xi)]^T.$$

Hence equation (1.9) shows that SA is symmetric, and, consequently, (1.1) is Friedrichs-symmetrizable when $d = 3$.

We now show that (1) $\Rightarrow$ (2).

Suppose that system (1.1) is Friedrichs-symmetrizable, i.e. there is a matrix S such that the product SA is symmetric.

We write S as a block matrix, compatible with A :

$$S := \begin{pmatrix} S_1 & \alpha & \beta \\ \alpha^T & S_2 & \gamma \\ \beta^T & \gamma^T & S_3 \end{pmatrix}.$$

We compute the product SA by block multiplication. Since SA is symmetric, for any block of SA , denoted $(SA)_{i,j}$ ($1 \leq i, j \leq 3$), we must have $(SA)_{i,j} = (SA)_{j,i}^T$. For $i = j = 3$, we obtain the constraint

$$\gamma^T C(\xi) + (v \cdot \xi) S_3 = [\gamma^T C(\xi) + (v \cdot \xi) S_3]^T.$$

Since S_3 is symmetric, we deduce that the product $\gamma^T C(\xi)$ is symmetric, for any $\xi \in \mathbb{R}^d$.

1. Symétrisation des équations moyennées de Reynolds – 2. Proof of the theorem

By computing explicitly the product, we obtain that the only possibility is that $\gamma = 0$.

For $i = 1$ and $j = 3$, we obtain the constraint

$$[\alpha C(\xi) + (v \cdot \xi)\beta]^T = (v \cdot \xi)\beta^T + \frac{1}{\rho}\gamma^T(p'(\rho)Id + P)\xi.$$

Since $\gamma = 0$, we obtain that $\alpha C(\xi) = 0$, for any $\xi \in \mathbb{R}^d$. By computing explicitly the product, we also obtain that $\alpha = 0$. Hence S has to be of the form

$$S = \begin{pmatrix} S_1 & 0 & \beta \\ 0 & S_2 & 0 \\ \beta^T & 0 & S_3 \end{pmatrix}.$$

For $i = 2$ and $j = 3$, we obtain the constraint

$$\rho\beta^T\xi^T + S_3D(\xi) = [S_2C(\xi)]^T \quad (1.13)$$

Solving the linear system (1.13) for S_2, S_3 and β gives after some computations that there exists two constants λ_1, λ_2 such that

$$S_2 = \lambda_1 P^{-1} \text{ and } \beta = \begin{cases} \lambda_2(q_{11}, 2q_{12}, q_{22}) & \text{when } d = 2, \\ \lambda_2(q_{11}, 2q_{12}, 2q_{13}, q_{22}, 2q_{23}, q_{33}) & \text{when } d = 3. \end{cases} \quad (1.14)$$

Note that it follows from these computations that $\beta D(\xi) = 2\lambda_2\xi^T$.

For $i = 1$ and $j = 2$ we obtain the constraint

$$\frac{1}{\rho}S_2(p'(\rho)Id + P)\xi = [S_1\rho\xi^T + \beta D(\xi)]^T = (S_1\rho + 2\lambda_2)\xi. \quad (1.15)$$

Equation (1.15) implies that S_2 is proportional to $(p'(\rho)Id + P)^{-1}$. Since S_2 is also proportional to P^{-1} by (1.14), we obtain that the two matrices P and $P + p'(\rho)Id$ are proportional (recall that S_2 is invertible). Hence $p' = 0$ or $P = \lambda Id$.

□

2. Étude de l'hyperbolicité de modèles d'écoulements diphasiques turbulents obtenus par principe variationnel

Dans ce chapitre, on présente une classe de systèmes hyperboliques modélisant des écoulements diphasiques à deux vitesses. Ces systèmes peuvent être obtenus par le principe d'action stationnaire de Hamilton, avec des contraintes conventionnelles : conservation de la masse pour chaque composante. Les équations impliquent la conservation de la quantité de mouvement totale et de l'énergie totale. L'hyperbolicité est garantie pour de petites vitesses relatives, i.e. près de l'équilibre, par la présence de turbulence prise en compte à l'aide d'un tenseur de Reynolds. L'incorporation de ce tenseur aux équations est faite via l'addition d'un terme turbulent dans l'énergie potentielle apparaissant dans le lagrangien du modèle. Plusieurs formes sont proposées pour le terme turbulent, et celles permettant d'obtenir un système d'équations hyperboliques sont caractérisées par un critère général. La présence du tenseur de Reynolds est cruciale pour l'hyperbolicité. En effet, s'il est retiré du lagrangien, les équations correspondantes ne sont pas hyperboliques lorsque la vitesse relative est petite. Ce chapitre est tiré d'un article publié dans la revue *Communications in Mathematical Sciences* (DELEAGE [2025](#)) :

Deléage, E. (2025). Hyperbolicity study of models for turbulent two-phase flows obtained from the variational principle. *Communications in Mathematical Sciences*, 23(6), 1631-1668

1. Introduction and main results

Multi-phase flow modeling aims to describe the simultaneous flow of different materials (solid, liquid, gas). It has a wide range of applications in biology, geosciences and industry, and is an active area of research at the intersection between physics, mechanics and mathematics. In this study, we focus on the case of two phases interacting with each other. We derive new two-phase flow models from Hamilton's principle, and we show that the presence of turbulence guarantees the hyperbolicity of the resulting equations.

1.1. Different multiphase flow models and hyperbolicity criterion

The multi-phase flow denomination can be used for very diverse phenomena occurring at various scales (Drew et al. 2006). Let us give some examples that can be found in nature. In geophysics, gravity driven flows such that rock, ice or snow avalanches and pyroclastic flows can be seen as multi-phase flows made of solid particles flowing in a surrounding gas (Chupin et al. 2024, Gray and Ancey 2011). Lava can also be modeled as a liquid-gas two phase flow (see the recent studies Narbona-Reina et al. 2024, Burgisser et al. 2024). In biology, the flow of human blood can be modeled as a suspension of red blood cells immersed in a viscous liquid (plasma) (Jung et al. 2006, Dong et al. 2023). Multi-phase flow theory is also used at the planetary scale in climate science, to model for instance density stratified oceans, or ocean-atmosphere couplings (see Vallis 2017).

In order to model these flows with accuracy, it is necessary to use models that take into account the specificity of each situation. The kind of model used usually depends on key features of the multi-phase flow, such as the nature of each component and the topology of the mixture. The microscopic description tracks the motion and interactions of individual particles. It is well-adapted to the study of granular suspensions when the number of solid particles is small (see for instance Lefebvre et al. 2007). When it is not possible to track each particle, a continuous description has to be used. There are two main kinds of continuous models for two-phase flows. When it is possible to identify distinct regions made of one phase and interacting with each other inside the flow, a separated two-phase system with interfaces can be used. This is the case for instance with the multi-layers models (see for instance Baines 2022). When the scale of these “one component regions” is much smaller than the typical scale of the motion, or when the two phases are dispersed, it is better to use an averaged description. In this kind of model, the two species are present at each space point, in a proportion given by volumes fractions. The dynamics is described by extended version of Euler or Navier-Stokes equations. Some models then use simplifying hypotheses in order to only keep one averaged velocity for all the phases (as in Chupin et al. 2024 for example). In other cases, the full complexity of the dynamics is taken into account and the velocity of each phase is a distinct unknown of the model. For instance, this is the case of the class of models presented in Coquel et al. 2014 that includes the Baer-Nunziato model.

In the following, we will focus on this last approach, but the three approaches are linked. Indeed, a way to derive a fully dispersed model is to start from a microscopic or a separated one, and to apply an averaging process. Closure relations are then needed in order to express averages of nonlinear and interfacial terms. An introduction to averaging techniques can be found in Drew et al. 2006. For the link between separated and dispersed phase flow models, see also Drui et al. 2019. For more recent work on the topic, see Bresch, Burtea, et al. 2022 and Bresch, Narbona-Reina, et al. 2024.

We shall now give a brief presentation of an important mathematical feature of reversible multi-phase continuum models, which is the hyperbolicity property. The

hyperbolicity property is an algebraic criterion that guarantees interesting behaviours when it is satisfied. From a mathematical point of view, it provides local well-posedness for the model. It means that, starting from a regular enough initial data, there exists a unique solution, which depends continuously on the initial data. From a physical point of view, the propagation of waves occurs at finite speed in hyperbolic systems, which is crucial for predictability and causality. For numerical purposes, hyperbolic systems can be solved by well-known and robust schemes. For an introduction to hyperbolic systems, see for instance the book of Benzoni-Gavage et al. 2006.

A priori, the hyperbolicity property is not satisfied in every multi-phase flow model (see below). In the framework of multi-velocity models, the hyperbolicity is often equivalent to a condition on the relative velocities, i.e. on the difference between the velocity of two different phases. In Saleh et al. 2020 for instance, it is shown that the hyperbolicity of a large class of barotropic systems is equivalent to the condition $|v_1 - v_k| \neq c_k, k = 2, \dots, N$, where N is the number of different phases, v_i is the velocity of the phase i and c_i is the sound speed of the phase i .

In the following, we will show that the introduction of turbulence in multi-phase flow models enables to guarantee that the hyperbolicity property is satisfied. Furthermore, the resulting models can be obtained from a variational principle.

1.2. Variational formulation of fluid mechanics

Let us now describe the method that will be used in order to derive the multi-phase models. It is called Hamilton's stationary action principle. This principle offers a unified way to formulate the governing evolution equations of many physical systems. It enables to obtain the governing evolution equations from a scalar function called the Lagrangian. The Lagrangian L is a function of the trajectory of the particles and is defined as the difference between the kinetic and potential energies of the system. Once the Lagrangian of the system is specified, one can define the associated action, which is the time integral of the Lagrangian. The stationary action principle then states that the particles follow a trajectory for which the action is stationary, eventually with additional constraints on the motion. The equations stating that the action is stationary characterize the trajectory and are called Euler-Lagrange equations.

Let us give some examples in the context of continuous media. In this case, one often defines a Lagrangian density $\mathcal{L}$, which corresponds to a Lagrangian per unit of volume. The Lagrangian L is then given by the integral of the Lagrangian density $\mathcal{L}$ over the whole domain. For a compressible barotropic fluid of density ρ and of velocity v , the Lagrangian density can be defined as

$$\mathcal{L} := \frac{1}{2} \rho |v|^2 - \rho \mathcal{E}(\rho).$$

The term $\rho |v|^2/2$ is the density of kinetic energy of the fluid, and the term $\rho \mathcal{E}(\rho)$ is called the density of internal energy. It is linked to the pressure of the fluid via the Maxwell relation $p(\rho) := \rho^2 \mathcal{E}'(\rho)$. The trajectory, denoted $x(t, X)$, and the velocity are

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 1. Introduction and main results

related by the following ordinary differential equation (ODE):

$$\partial_t x(t, X) = v(t, x(t, X)), \quad x(0, X) = X. \quad (2.1)$$

The capital letter X is used for the initial position of the particles, also called the Lagrangian coordinate. The letter x stands for the position of the particle in motion, also called the Eulerian coordinate. In order that $\mathcal{L}$ be a function of the trajectory only, one must specify how the density ρ depends on the trajectory. This is done by imposing the constraint of mass conservation, via the equation

$$\partial_t \rho + \operatorname{div}(\rho v) = 0. \quad (2.2)$$

With this constraint, one can apply the stationary action principle and obtain the corresponding Euler-Lagrange equation, which is an evolution equation for the velocity (see for instance Burtea, Gavriluk, et al. 2024 for detailed computations):

$$\partial_t v + (v \cdot \nabla) v + \frac{1}{\rho} \nabla p(\rho) = 0. \quad (2.3)$$

Equations (2.2) and (2.3) form together a well-known system called Euler compressible barotropic equations. This system is hyperbolic if and only if the pressure is an increasing function of the density, which amounts to require that the map $\rho \mapsto \rho \mathcal{E}(\rho)$ is convex. Other properties of this system are the conservation of momentum:

$$\partial_t(\rho v) + \operatorname{div}(\rho v \otimes v + p I_d)^T = 0,$$

and of total energy:

$$\partial_t e + \operatorname{div}[(e + p) v] = 0, \quad \text{with} \quad e := \frac{1}{2} \rho |v|^2 + \rho \mathcal{E}(\rho)$$

for smooth solutions. Hamilton's stationary action principle provides an additional interpretation of these conservation laws. Indeed, Noether's Theorem (see Noether 1918) states that conserved quantities correspond to symmetries of the Lagrangian. Here, the conservation of momentum comes from the spatial translation invariance of $\mathcal{L}$, and the conservation of energy is related to the invariance by time translation.

We now move on to the case of a two phase flow. The two phases will be denoted by the indices 1 and 2. The density, velocity and internal energy are respectively denoted by ρ_i , v_i and $\mathcal{E}_i$, $i = 1, 2$. Since the two velocities are different, it is also the case for the trajectories of the particles of the phase 1 and 2, denoted by x_i and defined as in (2.1) by

$$\partial_t x_i(t, X_i) = v_i(t, x_i(t, X_i)), \quad x_i(0, X_i) = X_i, \quad i = 1, 2.$$

The volume fraction of the phase i is denoted by α_i . The two volume fractions are linked by the constraint

$$\alpha_1 + \alpha_2 = 1, \quad (2.4)$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 1. Introduction and main results

which means that there is no vacuum. It is important to distinguish the true density ρ_i of each phase from the material density $\rho_{i,0} := \rho_i / \alpha_i$. The constraint of mass conservation only holds for the true densities:

$$\partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0, \quad i = 1, 2. \quad (2.5)$$

A possible choice to define the Lagrangian density of the mixture is to say that the kinetic (resp. internal) energy of the mixture is the sum of the kinetic (resp. internal) energy of each phase, weighted by the volume fractions:

$$\mathcal{L} = \frac{1}{2} \rho_1 |v_1|^2 + \frac{1}{2} \rho_2 |v_2|^2 - \rho_1 \mathcal{E}_1 \left(\frac{\rho_1}{\alpha_1} \right) - \rho_2 \mathcal{E}_2 \left(\frac{\rho_2}{\alpha_2} \right). \quad (2.6)$$

Physically, this means that there is no energy associated to the interactions between the two phases. Hamilton's principle under the constraints given by Equation (2.4) and Equations (2.5) yields a system of evolution equations for the variables $(\rho_1, v_1, \rho_2, v_2)$, for which the two pressures are equal ($p_1 = p_2$) and the volume fractions are given functions of the densities: $\alpha_i = \alpha_i(\rho_1, \rho_2)$. This system is hyperbolic if and only if the relative velocity is greater than an explicit lower bound: the system is hyperbolic when

$$|v_2 - v_1|^2 > \rho c_w^2 \left[\left(\frac{\alpha_1}{\rho_1} \right)^{1/3} + \left(\frac{\alpha_2}{\rho_2} \right)^{1/3} \right]^3,$$

where c_w is the speed of sound of the mixture and $\rho := \rho_1 + \rho_2$ is the total density (see Gavriluk 2011). This lower bound for the relative velocity can be problematic in situation where the two velocities are close, e.g. near equilibrium. Indeed, when the criterion is not fulfilled, complex characteristics appear.

In Gavriluk 2020, Hamilton's principle was again used together with the same Lagrangian (2.6) and the same constraints, and a different system was obtained, including an additional "lift" term. This counter-intuitive fact can be explain by the choice of the trajectories made by the author. Indeed, instead of defining the Lagrangian as a function of the trajectories of each phase x_1 and x_2 , the mean velocity of the fluid $v := (\rho_1 v_1 + \rho_2 v_2) / \rho$ (with $\rho = \rho_1 + \rho_2$ the total density) was used to define a third trajectory $x(t, X)$, given by the ODE $\partial_t x(t, X) = v(t, x(t, X))$. The Lagrangian was then considered as a function of the trajectories $x(t, X)$ and $x_2(t, X_2)$. As a consequence, the evolution equation obtained for the velocity v_i differs by an additional lift term equal to $\operatorname{curl}(v_1) \wedge (v - v_i)$ ($i = 1, 2$). The apparition of these lift terms is thus explained by the change of reference frame. Even if the obtained system is different, the hyperbolicity criterion remains unchanged for this modified model. In particular, the system is not hyperbolic when the norm of the relative velocity is small (Gavriluk 2020, Gavriluk 2011).

One possible interpretation of this lack of hyperbolicity is that some interaction terms are missing and should be added in the Lagrangian. In Gavriluk and Saurel 2002, an extension of the kinetic energy by addition of a term of the form $(d\alpha_1/dt)^2$ modeling inertia enabled to obtain an unconditional hyperbolicity for a model of

bubbly fluid (liquid-gas interaction). Allowing the Lagrangian to depend on derivatives of the unknowns is frequent in multi-fluid modeling. This is done for instance in Korteweg-type models, used in the context of liquid vapor mixture with a diffuse interface, and for which the Lagrangian depends on the gradient of the density (see e.g. Giesselmann et al. 2017).

1.3. Models studied and main results

In this work, we will show that two phase flow models that are hyperbolic for small relative velocity can be obtained from Hamilton's stationary action principle by adding an additional term to the Lagrangian, corresponding to a turbulent interaction. More precisely, we will consider the following Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{r}{2}\text{Tr}(P),$$

where $r \in \{\rho_1, \rho_2, \rho\}$ is either the density of one of the two phases, or the total density, depending on the choice of the model, and P is the Reynolds stress tensor. P is a second order symmetric definite positive tensor whose evolution equation is prescribed as an additional constraint to the model and is given by

$$\partial_t P + (u \cdot \nabla)P + \frac{\partial u}{\partial x}P + P\left(\frac{\partial u}{\partial x}\right)^T = 0, \quad (2.7)$$

where the velocity u drives the evolution of the tensor P and is given by $u \in \{v_1, v_2, v\}$ depending on the choice of the model.

The Reynolds stress tensor appears in the context of turbulent flows when a Reynolds averaging is performed. The tensor P is then obtained from the second order correlations of the velocity (see Pope 2001). Equation (2.7) is classical and appears whenever third order correlations are neglected (Mohammadi et al. 1993, Wilcox et al. 1998). It was shown in Teshukov 2007 that the same equation structure appears in the context of sheared shallow-water flows. In this context, the averaging operator is the depth-averaging. In the compressible barotropic case, Reynolds averaged equations form a system of hyperbolic equations (see Berthon, Coquel, et al. 2002). In the previous study of the author Deléage 2024, it was proved that this system of equations is symmetric hyperbolic if and only if the pressure is constant or the tensor P is scalar. The proof that Reynolds averaged equations can be obtained from Hamilton's stationary principle was presented in Gavriluk and Gouin 2020, in the case of a one phase flow.

Here, we aim to extend the Reynolds averaged equations to the case of a two phase flow. For two phases, there are various possibilities regarding the tensor P . Indeed, the turbulence can be carried either by one of the two phases, by the two phases, or by the mixture itself. We describe these different cases below.

Turbulence carried by the mixture velocity

Our first result concerns the case of a turbulence carried by the mean velocity. A possible interpretation is that the turbulence is caused by the mixture of the two phases. We thus first consider the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{\rho}{2}\text{Tr}(P), \quad (2.8)$$

with convex internal energies, and assume that the evolution equation of P is given by

$$\partial_t P + (v \cdot \nabla)P + \frac{\partial v}{\partial x}P + P\left(\frac{\partial v}{\partial x}\right)^T = 0, \quad (2.9)$$

with v the mean velocity, associated to the conservation of total density:

$$\partial_t \rho + \text{div}(\rho v) = 0$$

(this last equation being obtained by summing the conservation equation on ρ_1 and ρ_2). Assuming Equations (2.5) and (2.9) as constraints, the full system obtained by Hamilton's principle is given by (see Section 2 for the derivation):

$$\left\{ \begin{array}{ll} \partial_t \rho_i + \text{div}(\rho_i v_i) = 0, & i = 1, 2, \quad (2.10a) \\ \partial_t v_i + (v_i \cdot \nabla) v_i + \text{curl}(v_2) \wedge (v - v_i) + \nabla(U_i) + \frac{1}{\rho} \text{div}(\rho P)^T = 0, & i = 1, 2, \quad (2.10b) \\ \partial_t P + (v \cdot \nabla)P + \frac{\partial v}{\partial x}P + P\left(\frac{\partial v}{\partial x}\right)^T = 0. & (2.10c) \end{array} \right.$$

The pressure of the system is defined by

$$p = p_1 = p_2, \quad \text{with} \quad p_i := \left(\frac{\rho_i}{\alpha_i}\right)^2 \mathcal{E}'_i\left(\frac{\rho_i}{\alpha_i}\right), \quad i = 1, 2.$$

The relation $p_1 = p_2$ is the Euler-Lagrange equation associated to the variable α_1 . It enables to express the volume fractions α_1 and $\alpha_2 = 1 - \alpha_1$ as functions of the densities ρ_1, ρ_2 only via the implicit functions theorem. The functions U_i that appear in (2.10b) can then be expressed as

$$U_i(\rho_1, \rho_2) := \left(\frac{\partial}{\partial \rho_i} \left[\rho_i \mathcal{E}_i\left(\frac{\rho_i}{\alpha_i}\right) \right] \right) \Big|_{\alpha_i = \alpha_i(\rho_1, \rho_2)} \quad i = 1, 2. \quad (2.11)$$

As a consequence of the convexity of the internal energies $\rho_i \mathcal{E}_i\left(\frac{\rho_i}{\alpha_i}\right)$, the partial derivatives of the potentials U_i satisfy

$$a_1 := \frac{\partial U_1}{\partial \rho_1} > 0, \quad b := \frac{\partial U_1}{\partial \rho_2} = \frac{\partial U_2}{\partial \rho_1} > 0, \quad a_2 := \frac{\partial U_2}{\partial \rho_2} > 0, \quad (2.12)$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 1. Introduction and main results

as well as the additional relation

$$a_1 a_2 - b^2 = 0. \quad (2.13)$$

System (2.10) admits two additional conservation laws. For smooth solutions, the total momentum ρv is conserved:

$$\partial_t(\rho v) + \operatorname{div}(\rho_1 v_1 \otimes v_1 + \rho_2 v_2 \otimes v_2 + pId + \rho P)^T = 0, \quad (2.14)$$

as well as the total energy

$$\partial_t(e_1 + e_2) + \operatorname{div}(e_1 v_1 + e_2 v_2 + (\alpha_1 v_1 + \alpha_2 v_2)p + \rho P v)^T = 0. \quad (2.15)$$

The energy of each phase is given by

$$e_i := \frac{1}{2} \rho_i |v_i|^2 + \rho_i \mathcal{E}_i \left(\frac{\rho_i}{\alpha_i} \right) + \frac{1}{2} \rho_i \operatorname{Tr} P, \quad i = 1, 2,$$

Once again, the variational formulation of System (2.10) enables to relate the conservation laws with symmetries of the Lagrangian density (2.8). Indeed, Noether's theorem states that the conservation of momentum (2.14) is associated to the invariance of (2.8) by spatial translations, while the conservation of energy (2.15) is associated to the invariance by time translation.

Note the presence of the lift term $\operatorname{curl}(v_2) \wedge (v - v_i)$ in the evolution equation of v_i . As explained before, this term appear because the Lagrangian was considered as a function of the two trajectories $x_1(t, X_1)$ associated to v_1 and $x(t, X)$ associated to the mean velocity v . The choice of the trajectory $x(t, X)$ is inevitable in the derivation of the equations in order to be able to express the constraint (2.9) in Lagrangian coordinates (see the detailed derivation in Section 2). Since the lift terms depend on the choice of the trajectories and are uncommon in the literature, we also study a simpler version of System (2.10), where the lift terms are removed:

$$\left\{ \begin{array}{ll} \partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0, & i = 1, 2, \quad (2.16a) \\ \partial_t v_i + (v_i \cdot \nabla v_i) + \nabla(U_i) + \frac{1}{\rho} \operatorname{div}(\rho P)^T = 0, & i = 1, 2, \quad (2.16b) \\ \partial_t P + (v \cdot \nabla) P + \frac{\partial v}{\partial x} P + P \left(\frac{\partial v}{\partial x} \right)^T = 0. & (2.16c) \end{array} \right.$$

The hyperbolicity of Systems (2.10) and (2.16) is characterized by the following theorem:

Theorem 2.1. *(Hyperbolicity of two-phase flow with a turbulence carried by the mean velocity)*

Let us denote $\rho_{i,0} := \rho_i / \alpha_i$ the material density of each fluid, for $i = 1, 2$. Let also

$$\theta := \rho_1 \rho_2 (\sqrt{a_1} - \sqrt{a_2})^2 - \frac{1}{2} (\rho_1 \sqrt{a_1} + \rho_2 \sqrt{a_2})^2, \quad (2.17)$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 1. Introduction and main results

a_1, a_2 being defined in (2.12). Then it holds that:

- In dimension $d = 1$, the lift terms of Equation (2.10b) vanish and System (2.10) is hyperbolic whenever $\rho_{1,0} \neq \rho_{2,0}$, for small relative velocity. In other words, there exists a constant $C = C(\rho_1, \rho_2, P) > 0$ such that (2.10) is hyperbolic whenever $|v_1 - v_2| < C$.
- When $d = 2$, suppose again that $\rho_{1,0} \neq \rho_{2,0}$. Also suppose that

$$|v_1 - v_2| < C, \quad \text{and either } \rho P > \theta I_d \quad \text{or} \quad \rho P < \theta I_d. \quad (2.18)$$

Then both System (2.10) and System (2.16) are hyperbolic.

- When $d = 3$, system (2.16) is hyperbolic under the assumptions $\rho_{1,0} \neq \rho_{2,0}$ and (2.18). Under these two assumptions, System (2.10) is only weakly hyperbolic.

In dimension 2 and 3, if the condition $\rho P > \theta I_d$ or $\rho P < \theta I_d$ is not satisfied, System (2.10) and (2.16) are only weakly hyperbolic.

The fact that the hyperbolicity of the system is improved after removing the lift terms is quite surprising and is a consequence of the presence of the Reynolds stress tensor. Indeed, as already explained in the end of paragraph 1.2, the hyperbolicity criterion is not affected by the addition or removal of lift terms for a two-phase flow without Reynolds stress tensor. See also Remark 2.2 below.

Turbulence carried by one of the two phases

We now suppose that the turbulence is only carried by one phase, say the phase 2. We thus consider the following Lagrangian :

$$\mathcal{L} := \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{1}{2}\rho_2\text{Tr}(P_2). \quad (2.19)$$

We use the superscript 2 for the tensor P_2 to insist on the fact that it is carried by the phase two. As a consequence, we assume that the evolution of P_2 is given by

$$\partial_t P_2 + (v_2 \cdot \nabla) P_2 + \frac{\partial v_2}{\partial x} P_2 + P_2 \left(\frac{\partial v_2}{\partial x} \right)^T = 0. \quad (2.20)$$

We also impose the conservation of the two densities:

$$\partial_t \rho_i + \text{div}(\rho_i v_i) = 0, \quad i = 1, 2. \quad (2.21)$$

In order to lighten the presentation, the system obtained is given by Equation (2.52) of Section 4. The result concerning this system is the following:

Theorem 2.2. *(Hyperbolicity of two-phase flow with a turbulence carried by one phase only)*

The hyperbolicity of the system obtained by Hamilton's principle from the Lagrangian (2.19) and the constraints (2.20) and (2.21) can be characterized as follows:

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 1. Introduction and main results

- When $d = 1$, System (2.52) is hyperbolic for small relative velocity $|v_1 - v_2|$, i.e. there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that (2.52) is hyperbolic when $|v_1 - v_2| \leq C$.
- When $d = 2, 3$ let

$$\mu := \rho_1 a_1 - \frac{1}{2} \rho_2 a_2$$

and assume that either $P_2 > \mu I_d$ or $P_2 < \mu I_d$. Then (2.52) is hyperbolic for $|v_1 - v_2| \leq C$. If the condition $P_2 > \mu I_d$ or $P_2 < \mu I_d$ is not satisfied, System (2.52) is only weakly hyperbolic.

Turbulence carried by each of the two phases

We also study the case of the mixture of two turbulent phases. In other words, each phase is endowed with a Reynolds stress tensor denoted P_i , $i = 1, 2$. The Lagrangian density is the following:

$$\mathcal{L} := \frac{1}{2} \rho_1 |v_1|^2 + \frac{1}{2} \rho_2 |v_2|^2 - \rho_1 \mathcal{E}_1 \left(\frac{\rho_1}{\alpha_1} \right) - \rho_2 \mathcal{E}_2 \left(\frac{\rho_2}{\alpha_2} \right) - \frac{1}{2} \rho_1 \text{Tr}(P_1) - \frac{1}{2} \rho_2 \text{Tr}(P_2). \quad (2.22)$$

The densities are still conserved:

$$\partial_t \rho_i + \text{div}(\rho_i v_i) = 0, \quad i = 1, 2.$$

The evolution equations for P_1, P_2 are given by

$$\partial_t P_i + (v_i \cdot \nabla) P_i + \frac{\partial v_i}{\partial x} P_i + P_i \left(\frac{\partial v_i}{\partial x} \right)^T = 0, \quad i = 1, 2. \quad (2.23)$$

In order to lighten the presentation, the system obtained is given by Equation (2.54) of Section 4. The result concerning this system is the following:

Theorem 2.3. (Hyperbolicity of the mixture of two turbulent phases)

The hyperbolicity of the system obtained by Hamilton's principle from the Lagrangian (2.22) under the constraints of mass conservation and (2.23) can be characterized as follows:

- When $d = 1$, System (2.54) is hyperbolic for small relative velocity $|v_1 - v_2|$, i.e. there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that (2.54) is hyperbolic when $|v_1 - v_2| \leq C$.
- When $d = 2, 3$ assume that

$$\frac{1}{3} P_1 \leq P_2 \leq 3 P_1, \quad \text{or equivalently} \quad \frac{1}{3} P_2 \leq P_1 \leq 3 P_2. \quad (2.24)$$

Then (2.54) is hyperbolic whenever $|v_1 - v_2| \leq C$. If (2.24) is not satisfied, (2.54) is only weakly hyperbolic.

General criterion of hyperbolicity for turbulent two-phase flows

We finally come back to the general Lagrangian density

$$\mathcal{L} = \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{r}{2}\text{Tr}(P),$$

The three previous results show that the turbulence $r\text{Tr}P/2$ can be added in various ways to the potential energy of a two-phase flow in order to obtain a hyperbolic system of equations. The next and last result states that the three examples given previously are the only ones for which the resulting system of equations is hyperbolic.

Theorem 2.4. *(General criterion of hyperbolicity for turbulent two-phase flows)*

Consider the following general Lagrangian density:

$$\mathcal{L} := \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{1}{2}r\text{Tr}(P), \quad (2.25)$$

endowed with the constraints

$$\partial_t \rho_i + \text{div}(\rho_i v_i) = 0, \quad i = 1, 2 \quad (2.26)$$

and

$$\partial_t P + (u \cdot \nabla)P + \frac{\partial u}{\partial x}P + P\left(\frac{\partial u}{\partial x}\right)^T = 0, \quad (2.27)$$

where

$$(r, u) \in \{\rho_1, \rho_2, \rho\} \times \{v_1, v_2, v\}.$$

Let us consider the system formed by Equations (2.26), (2.27) and the Euler-Lagrange equations associated to the Lagrangian density (2.25). We denote by M the matrix of this system. Then, in dimension 1, all the eigenvalues of M are real when $v_1 = v_2$, whatever the choice of convex internal energies $\rho_i\mathcal{E}_i(\rho)$, $i = 1, 2$, if and only if the density r and the velocity u are compatible, i.e. the following equation is satisfied:

$$\partial_t r + \text{div}(ru) = 0.$$

Let us give two remarks to complete the results obtained.

Remark 2.1. *When one of the two phases, say the phase 1, is made of solid particles, one often assumes that the material density of the phase 1, given by*

$$\rho_{1,0} := \frac{\rho_1}{\alpha_1}$$

is a constant. This is the case for instance for a dry granular flow or for a granular suspension. Note that it is then straightforward to express the volume fractions as

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 1. Introduction and main results

functions of the density ρ_1 :

$$\alpha_1 = \frac{\rho_1}{\rho_{1,0}}, \quad \alpha_2 = 1 - \frac{\rho_1}{\rho_{1,0}}.$$

All the systems and hyperbolicity results obtained are still valid under the following modifications:

- There is no internal energy associated to the phase 1: $\mathcal{E}_1 = 0$.
- The pressure is defined by

$$p = p_1 = p_2 = \left(\frac{\rho_2}{\alpha_2} \right)^2 \mathcal{E}_2' \left(\frac{\rho_2}{\alpha_2} \right),$$

- The chemical potentials U_i are defined by

$$U_i(\rho_1, \rho_2) := \frac{\partial}{\partial \rho_i} \left[\rho_2 \mathcal{E}_2 \left(\frac{\rho_2}{1 - \rho_1 / \rho_{1,0}} \right) \right], \quad i = 1, 2.$$

Remark 2.2. In dimension 2 or 3, the results obtained state that the hyperbolicity is valid under additional conditions on the spectrum of the Reynolds tensor (see (2.18) for instance). These additional conditions are necessary to prevent some eigenvalues of the systems to appear with a multiplicity greater than 1. They are called non resonance conditions. If they are not satisfied, the systems obtained are only weakly hyperbolic. The solution of weakly hyperbolic systems are known to be less regular than the ones of hyperbolic systems. In particular, phenomena like creation of vacuum or high concentration (delta shocks) can occur (see Mitrović et al. 2007 for instance).

Recall that the Reynolds stress tensor is a symmetric, definite positive tensor. As a consequence, in some cases, these non resonance conditions are always satisfied. For instance, the condition $\rho P > \theta I_d$ or $\rho P < \theta I_d$ of Theorem 2.1 is automatically satisfied when $\theta < 0$, which is the case whenever a_1 is close to a_2 . Similarly, the non resonance condition of Theorem 2.2 is automatically satisfied whenever $\mu < 0$.

Outline of the paper

The rest of this document is organised as follows. In Section 2, we give the derivation of the two-phase flow system with a mixture turbulence, System (2.10), from Hamilton's principle. In Section 3, we give a detailed study of the hyperbolicities of this system and its simplified version, System (2.16), in order to prove Theorem 2.1. In Section 4, we show the hyperbolicity of the two other models presented in the introduction (Theorems 2.2 and 2.3) and we establish the general hyperbolicity criterion stated in Theorem 2.4.

2. Derivation of the model from variational principle

Notations

The space dimension is denoted d and can be equal to 1, 2 or 3. We denote $\partial_i := \partial/\partial x_i$ the partial derivative of the variable x_i , for $1 \leq i \leq d$. If $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a scalar function, we denote by $\nabla f \in \mathbb{R}^d$ the gradient of f , i.e. the vector field of components $\partial_i f$, $1 \leq i \leq d$.

If $Z = (Z_1, \dots, Z_d) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a vector field, the divergence of Z is the scalar function defined by

$$\operatorname{div}(Z) := \partial_1 Z_1 + \dots + \partial_d Z_d.$$

We also denote by $\partial Z / \partial x$ the Jacobian matrix of Z , i.e. the matrix of coefficients $(\partial Z / \partial x)_{i,j} = \partial Z_i / \partial x_j$, for $1 \leq i, j \leq d$.

If $Z = (Z_i)_{1 \leq i \leq d}$ and $Z' = (Z'_i)_{1 \leq i \leq d}$ are two vector fields, we denote $Z \otimes Z'$ the second order tensor defined by $Z \otimes Z' := Z(Z')^T$, i.e. the matrix of coefficients $(Z \otimes Z')_{i,j} = Z_i Z'_j$, for $1 \leq i, j \leq d$.

If $A : \mathbb{R}^d \rightarrow M_d(\mathbb{R})$ is a second order tensor, we define the divergence of A as the line vector of $\mathbb{R}^d$ whose i -th component is given by the divergence of the i -th column of A .

We denote I_d the identity matrix. For any positive integer d , we denote $S_d^{++}(\mathbb{R})$ the set of symmetric definite positive matrices, i.e. the symmetric matrices of size d with a positive spectrum.

Derivation of System (2.10) from variational principle

We now give the derivation of System (2.10) from Hamilton's stationary action principle. Various arguments that are used in this proof can be found in other works. We refer to Gavriluk and Gouin 2020 for the derivation of Reynolds averaged equations via variational principle. Detailed computations of Euler-Lagrange equations in different reference frames were performed in Gavriluk 2020. Finally, we refer to Burtea, Gavriluk, et al. 2024 for an introduction to the application of Hamilton's principle to fluid mechanics.

We consider a two-phase flow with a Lagrangian density given by

$$\mathcal{L} = \frac{1}{2} \rho_1 |v_1|^2 + \frac{1}{2} \rho_2 |v_2|^2 - \rho_1 \mathcal{E}_1 \left(\frac{\rho_1}{\alpha_1} \right) - \rho_2 \mathcal{E}_2 \left(\frac{\rho_2}{\alpha_2} \right) - \frac{\rho}{2} \operatorname{Tr}(P), \quad (2.28)$$

where ρ_i (resp. v_i) is the density (resp. velocity) of the phase i , $i = 1, 2$. The two density are conserved :

$$\partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0, \quad i = 1, 2, \quad (2.29)$$

which implies that the total density $\rho := \rho_1 + \rho_2$ is also conserved:

$$\partial_t \rho + \operatorname{div}(\rho v) = 0, \quad \text{with} \quad \rho v := \rho_1 v_1 + \rho_2 v_2.$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 2. Derivation of the model from variational principle

α_1 and α_2 are the volume fractions of the phases 1 and 2 and verify the constraint

$$\alpha_1 + \alpha_2 = 1,$$

hence $\alpha_2 = 1 - \alpha_1$. The evolution of the Reynolds stress tensor is given by

$$\partial_t P + (v \cdot \nabla) P + \frac{\partial v}{\partial x} P + P \left(\frac{\partial v}{\partial x} \right)^T = 0. \quad (2.30)$$

We aim to show that Hamilton's principle applied to the Lagrangian (2.28) endowed with the two constraints (2.29) and (2.30) yields Equation (2.10b) as the Euler-Lagrange equation associated to the velocity v_i :

$$\partial_t v_i + (v_i \cdot \nabla) v_i + \text{curl}(v_2) \wedge (v - v_i) + \nabla(U_i) + \frac{1}{\rho} \text{div}(\rho P)^T = 0, \quad i = 1, 2.$$

2.1. Trajectories and Lagrangian quantities

Let us suppose that the two-phase flow lies in a fixed open set $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$. We denote by $X \in \Omega$ (or X_i , $i = 1, 2$) the initial position of an element of fluid at time $t = 0$. X is usually called the Lagrangian coordinate. The trajectory of an element of the phase 1, flowing at velocity v_1 , that started at the point X_1 at time $t = 0$, is denoted by $x_1(t, X_1) \in \Omega$. It is defined by

$$\partial_t x_1(t, X_1) = v_1(t, x_1(t, X_1)), \quad \text{with} \quad x_1(0, X_1) = X_1.$$

We will assume that $x_1(t) : \Omega \rightarrow \Omega$ is a diffeomorphism for every $t \in [0, T]$, where $T > 0$ is fixed, and we will denote by $x_1(t)^{-1}$ its inverse. The trajectory x_1 enables to define Lagrangian quantities, i.e. quantities that depend on the Lagrangian coordinates. Three examples are the Lagrangian velocity v_1^l , the Lagrangian density ρ_1^l and the Lagrangian volume fraction α_1^l , defined by

$$v_1^l(t, X_1) := v_1(t, x_1(t, X_1)), \quad \rho_1^l(t, X_1) := \rho_1(t, x_1(t, X_1)), \quad \alpha_1^l(t, X_1) := \alpha_1(t, x_1(t, X_1)).$$

From the chain rule and the mass conservation, one then computes that

$$\partial_t \rho_1^l(t, X_1) = (\partial_t \rho_1 + v_1 \cdot \nabla \rho_1)(t, x_1(t, X_1)) = -[\rho_1 \text{div}(v_1)](t, x_1(t, X_1)). \quad (2.31)$$

Using the identity $v_1 = v_1^l \circ x_1(t)^{-1}$ and the chain rule once again,

$$\begin{aligned} \text{div}(v_1) &= \text{Tr} \left(\frac{\partial v_1}{\partial x} \right) = \text{Tr} \left[\left(\frac{\partial v_1^l}{\partial X_1} \circ x_1(t)^{-1} \right) \left(\frac{\partial x_1(t)^{-1}}{\partial x} \right) \right] \\ &= \text{Tr} \left[\left(\frac{\partial v_1^l}{\partial X_1} \right) \left(\frac{\partial x_1}{\partial X_1} \right)^{-1} \right] \circ x_1(t)^{-1} = \frac{\partial_t J_1}{J_1} \circ x_1(t)^{-1}, \end{aligned}$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 2. Derivation of the model from variational principle

where we defined

$$J_1 := \det \left(\frac{\partial x_1}{\partial X_1} \right)$$

the Jacobian of the trajectory x_1 and we used the identity

$$\frac{d}{ds} [\det A(s)] = \det A(s) \operatorname{Tr} \left[\frac{dA}{ds} A(s)^{-1} \right], \quad (2.32)$$

valid for any C^1 function $s \mapsto A(s)$ such that the square matrix $A(s)$ is invertible. Returning to (2.31), we obtain

$$\partial_t \rho_1^l = -\rho_1^l \frac{\partial_t J_1}{J_1}, \quad \text{i.e.} \quad \partial_t (\rho_1^l J_1) = 0, \quad \text{hence} \quad \rho_1^l = \frac{\rho_1|_{t=0}}{J_1} \quad (2.33)$$

(note that $x_1(0) = Id$, hence $J_1|_{t=0} = 1$). We deduce that the density ρ_1 is determined by its value at time $t = 0$ and by the trajectory x_1 .

In a similar way, we define $x(t, X)$ the solution of

$$\partial_t x(t, X) = v(t, x(t, X)), \quad x(0, X) = X.$$

In other words, $x(t, X)$ is the trajectory of a particle flowing at the mean velocity that started in position X at time 0. As before, one can define the Lagrangian mean velocity, the Lagrangian total density and the Lagrangian Reynolds tensor by

$$v^l(t, X) := v(t, x(t, X)), \quad \rho^l(t, X) := \rho(t, x(t, X)) \quad \text{and} \quad P^l(t, X) := P(t, x(t, X)).$$

As in Equation (2.33), we deduce that

$$\rho^l = \frac{\rho^l|_{t=0}}{J}, \quad \text{with} \quad J := \det \left(\frac{\partial x}{\partial X} \right). \quad (2.34)$$

Furthermore, the chain rule yields that P^l solves an ordinary differential equation, given by

$$\partial_t P^l + \left(\frac{\partial v}{\partial x} \circ x(t) \right) P^l + P^l \left(\frac{\partial v}{\partial x} \circ x(t) \right)^T = 0.$$

Hence P is also uniquely determined by the trajectory and the initial value $P|_{t=0}$.

2.2. Action and variations

Recall that the Lagrangian density of System (2.10) was given in Equation (2.8) by

$$\mathcal{L} = \frac{1}{2} \rho_1 |v_1|^2 + \frac{1}{2} \rho_2 |v_2|^2 - \rho_1 \mathcal{E}_1 \left(\frac{\rho_1}{\alpha_1} \right) - \rho_2 \mathcal{E}_2 \left(\frac{\rho_2}{\alpha_2} \right) - \frac{1}{2} \rho \operatorname{Tr}(P).$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 2. Derivation of the model from variational principle

This Lagrangian density can also be written in terms of the variables $(\rho_1, \rho, v_1, v, \alpha_1, \rho \text{Tr} P)$:

$$\mathcal{L}(\rho_1, \rho, v_1, v, \alpha_1, \rho \text{Tr} P) := \frac{1}{2} \rho |v|^2 + \frac{\rho_1 \rho}{2(\rho - \rho_1)} |v - v_1|^2 - U(\rho_1, \rho - \rho_1, \alpha_1) - \frac{1}{2} \rho \text{Tr} P,$$

where

$$U(\rho_1, \rho_2, \alpha_1) := \rho_1 \mathcal{E}_1 \left(\frac{\rho_1}{\alpha_1} \right) + \rho_2 \mathcal{E}_2 \left(\frac{\rho_2}{1 - \alpha_1} \right),$$

From now on, we will see $\mathcal{L}$ as a function of the variables $(\rho_1, \rho, v_1, v, \alpha_1, \rho \text{Tr} P)$ only. One can now define the action associated to this Lagrangian:

$$\mathcal{A} [x_1, x, \alpha_1^l] := \int_0^T \int_{\Omega} \mathcal{L}(\rho_1, \rho, v_1, v, \alpha_1, \rho \text{Tr} P) dx dt.$$

Let $\Phi = (\phi_1, \phi, \phi_{\alpha_1}) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ be a smooth vector field which is compactly supported in $(0, T) \times \Omega$. For every $\epsilon \in \mathbb{R}$, one can perturb the trajectories and define $(x_{1,\epsilon}, x_\epsilon, \alpha_{1,\epsilon}^l) := (x_1, x, \alpha_1^l) + \epsilon \Phi$. To these modified trajectories, one can associate the corresponding densities, velocities and Reynolds stress tensor that we will denote $\rho_{1,\epsilon}, \rho_\epsilon, v_{1,\epsilon}, v_\epsilon, P_\epsilon$. We define

$$\mathcal{A}_\epsilon := \mathcal{A} [x_{1,\epsilon}, x_\epsilon, \alpha_{1,\epsilon}^l]$$

the perturbed action. Hamilton's stationary action principle states that the trajectory followed by the particles is a stationary point of the action, i.e. that

$$\left. \frac{\partial \mathcal{A}_\epsilon}{\partial \epsilon} \right|_{\epsilon=0} = 0,$$

for every perturbation Φ . In order to simplify notations, we denote, for every quantity f_ϵ depending on ϵ , $\delta f := (\partial_\epsilon f_\epsilon)|_{\epsilon=0}$. δf is often called the variation of f . The stationary action principle is thus equivalent to require that the variation of the action is equal to zero: $\delta \mathcal{A} = 0$. We will also define $\zeta_1 := \phi_1 \circ x_1(t)^{-1}$, $\zeta := \phi \circ x(t)^{-1}$ and $\zeta_{\alpha_1} := \phi_{\alpha_1} \circ x_1(t)^{-1}$. From the stationary action principle, one obtains the following equality:

$$0 = \delta \mathcal{A} = \int_0^T \int_{\Omega} \left[\frac{\partial \mathcal{L}}{\partial \rho_1} \delta \rho_1 + \frac{\partial \mathcal{L}}{\partial \rho} \delta \rho + \frac{\partial \mathcal{L}}{\partial v_1} \delta v_1 + \frac{\partial \mathcal{L}}{\partial v} \delta v + \frac{\partial \mathcal{L}}{\partial \alpha_1} \delta \alpha_1 + \frac{\partial \mathcal{L}}{\partial (\rho \text{Tr} P)} \delta (\rho \text{Tr} P) \right] dx dt.$$

We thus need to compute the variations of all variables. First, it follows from the conservation of mass (2.33) and (2.34), as well as the formula (2.32) that

$$\delta \rho_1 = -\text{div}(\rho_1 \zeta_1) \quad \text{and} \quad \delta \rho = -\text{div}(\rho \zeta).$$

As a consequence, we can integrate by parts to obtain that

$$\int_0^T \int_{\Omega} \frac{\partial \mathcal{L}}{\partial \rho_1} \delta \rho_1 = \int_0^T \int_{\Omega} \nabla \left(\frac{\partial \mathcal{L}}{\partial \rho_1} \right) \cdot \rho_1 \zeta_1 \quad \text{and} \quad \int_0^T \int_{\Omega} \frac{\partial \mathcal{L}}{\partial \rho} \delta \rho = \int_0^T \int_{\Omega} \nabla \left(\frac{\partial \mathcal{L}}{\partial \rho} \right) \cdot \rho \zeta.$$

We now move on to the variation of the velocities. Since $v_1 \circ x_1(t) = v_1^l$ and $\delta v_1^l = \partial_t \phi_1$,

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 2. Derivation of the model from variational principle

we get that

$$\delta v_1 = \partial_t \zeta_1 + \frac{\partial \zeta_1}{\partial x} v_1 - \frac{\partial v_1}{\partial x} \zeta_1.$$

Let us define

$$K_1 := \frac{1}{\rho_1} \left(\frac{\partial \mathcal{L}}{\partial v_1} \right)^T, \quad \text{hence} \quad \rho_1 K_1^T = \frac{\partial \mathcal{L}}{\partial v_1}.$$

An integration by part coupled by the conservation of ρ_1 yields

$$\begin{aligned} \int_0^T \int_{\Omega} \frac{\partial \mathcal{L}}{\partial v_1} \delta v_1 &= \int_0^T \int_{\Omega} \rho_1 K_1 \cdot \left(\partial_t \zeta_1 + \frac{\partial \zeta_1}{\partial x} v_1 - \frac{\partial v_1}{\partial x} \zeta_1 \right) \\ &= - \int_0^T \int_{\Omega} \rho_1 \zeta_1 \cdot \left(\partial_t K_1 + \frac{\partial K_1}{\partial x} v_1 + \left(\frac{\partial v_1}{\partial x} \right)^T K_1 \right). \end{aligned}$$

In a similar way, we define

$$K := \frac{1}{\rho} \left(\frac{\partial \mathcal{L}}{\partial v} \right)^T$$

and obtain that

$$\int_0^T \int_{\Omega} \frac{\partial \mathcal{L}}{\partial v} \delta v = - \int_0^T \int_{\Omega} \rho \zeta \cdot \left(\partial_t K + \frac{\partial K}{\partial x} v + \left(\frac{\partial v}{\partial x} \right)^T K \right).$$

Concerning the variations of α_1 , we compute that

$$\delta \alpha_1 = \zeta_{\alpha_1} - \frac{\partial \alpha_1}{\partial x} \zeta_1.$$

Finally, the equation satisfied by P implies that the variation of P is given by

$$\delta \text{Tr} P = -2 \text{Tr} \left(\frac{\partial \zeta}{\partial x} P \right) - \frac{\partial (\text{Tr} P)}{\partial x} \zeta$$

(see Gavriluk and Gouin 2020). In particular, the field ζ is needed to express the variations of P . This is why we use the trajectory $x(t, X)$ corresponding to the mean velocity v and responsible for the presence of lift terms (see Gavriluk 2020 and below). It follows that

$$\delta(\rho \text{Tr} P) = -2 \text{Tr} \left(\frac{\partial \zeta}{\partial x} \rho P \right) - \text{div}(\rho \text{Tr} P \zeta).$$

Hence, by integration by parts,

$$\int_0^T \int_{\mathbb{R}} \frac{\partial \mathcal{L}}{\partial(\rho \text{Tr} P)} \delta(\rho \text{Tr} P) = \int_0^T \int_{\mathbb{R}} \text{Tr} \left(\frac{\partial \zeta}{\partial x} \rho P \right) = - \int_0^T \int_{\mathbb{R}} \text{div}(\rho P) \zeta.$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 2. Derivation of the model from variational principle

The stationary action principle can thus be written

$$\begin{aligned} 0 = \delta \mathcal{A} = & - \int_0^T \int_{\Omega} \rho_1 \zeta_1 \cdot \left(\partial_t K_1 + \frac{\partial K_1}{\partial x} v_1 + \left(\frac{\partial v_1}{\partial x} \right)^T K_1 - \nabla \left(\frac{\partial \mathcal{L}}{\partial \rho_1} \right) + \frac{1}{\rho_1} \frac{\partial \alpha_1}{\partial x} \frac{\partial \mathcal{L}}{\partial \alpha_1} \right) \\ & - \int_0^T \int_{\Omega} \rho \zeta \cdot \left(\partial_t K + \frac{\partial K}{\partial x} v + \left(\frac{\partial v}{\partial x} \right)^T K - \nabla \left(\frac{\partial \mathcal{L}}{\partial \rho} \right) + \frac{1}{\rho} \operatorname{div}(\rho P)^T \right) \\ & + \int_0^T \int_{\Omega} \frac{\partial \mathcal{L}}{\partial \alpha_1} \zeta_{\alpha_1}. \end{aligned}$$

Since ζ_1, ζ and ζ_{α_1} are arbitrary and independent of each other, each of the integrands must vanish and we obtain the following Euler-Lagrange equations:

$$0 = \partial_t K_1 + \frac{\partial K_1}{\partial x} v_1 + \left(\frac{\partial v_1}{\partial x} \right)^T K_1 - \nabla \left(\frac{\partial \mathcal{L}}{\partial \rho_1} \right) + \frac{1}{\rho_1} \frac{\partial \alpha_1}{\partial x} \frac{\partial \mathcal{L}}{\partial \alpha_1}, \quad (2.35)$$

$$0 = \partial_t K + \frac{\partial K}{\partial x} v + \left(\frac{\partial v}{\partial x} \right)^T K - \nabla \left(\frac{\partial \mathcal{L}}{\partial \rho} \right) + \frac{1}{\rho} \operatorname{div}(\rho P)^T, \quad (2.36)$$

$$0 = \frac{\partial \mathcal{L}}{\partial \alpha_1}. \quad (2.37)$$

2.3. Computation of the equation of the velocities

We can now use the three Euler-Lagrange equations obtained to get the evolution equations for the velocities v_i , $i = 1, 2$. Note that Equation (2.35) can be simplified with the help of Equation (2.37). We also compute that

$$\begin{aligned} K_1 &= v_1 - v_2, & \frac{\partial \mathcal{L}}{\partial \rho_1} &= \frac{1}{2} |v_1 - v_2|^2 - \frac{\partial U}{\partial \rho_1} + \frac{\partial U}{\partial \rho_2}, \\ K &= v_2, & \frac{\partial \mathcal{L}}{\partial \rho} &= \frac{1}{2} |v|^2 - \frac{1}{2} |v - v_2|^2 - \frac{\partial U}{\partial \rho_2}. \end{aligned}$$

Hence after adding (2.36) to (2.35), we obtain the equation for v_1 :

$$\partial_t v_1 + (v_1 \cdot \nabla) v_1 + \operatorname{curl}(v_2) \wedge (v - v_1) + \nabla \left(\frac{\partial U}{\partial \rho_1} \right) + \frac{1}{\rho} \operatorname{div}(\rho P)^T = 0.$$

We also compute that (2.36) yields the equation for v_2 :

$$\partial_t v_2 + (v_2 \cdot \nabla) v_2 + \operatorname{curl}(v_2) \wedge (v - v_2) + \nabla \left(\frac{\partial U}{\partial \rho_2} \right) + \frac{1}{\rho} \operatorname{div}(\rho P)^T = 0.$$

We still need to express these two equations without the dependence on α_1 coming from $(\partial U / \partial \rho_i)$. In order to do this, we define $g(\rho_1, \rho_2, \alpha_1) := \partial U / \partial \alpha_1$ and obtain with Equation (2.37) that

$$g(\rho_1, \rho_2, \alpha_1) = \frac{\partial U}{\partial \alpha_1} = - \frac{\partial \mathcal{L}}{\partial \alpha_1} = 0.$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 2. Derivation of the model from variational principle

We compute that

$$g(\rho_1, \rho_2, \alpha_1) = \frac{\partial U}{\partial \alpha_1} = -\left(\frac{\rho_1}{\alpha_1}\right)^2 \mathcal{E}'_1\left(\frac{\rho_1}{\alpha_1}\right) + \left(\frac{\rho_2}{1-\alpha_1}\right)^2 \mathcal{E}'_2\left(\frac{\rho_2}{1-\alpha_1}\right) = p_2 - p_1,$$

hence (2.37) states that the two pressures are equal. Furthermore,

$$\begin{aligned} \frac{\partial g}{\partial \alpha_1} &= \frac{\partial^2 U}{\partial \alpha_1^2} = \frac{1}{\alpha_1} \left[2\left(\frac{\rho_1}{\alpha_1}\right)^2 \mathcal{E}'_1\left(\frac{\rho_1}{\alpha_1}\right) + \left(\frac{\rho_1}{\alpha_1}\right)^3 \mathcal{E}_1^{(2)}\left(\frac{\rho_1}{\alpha_1}\right) \right] \\ &\quad + \frac{1}{1-\alpha_1} \left[2\left(\frac{\rho_2}{1-\alpha_1}\right)^2 \mathcal{E}'_2\left(\frac{\rho_2}{1-\alpha_1}\right) + \left(\frac{\rho_2}{1-\alpha_1}\right)^3 \mathcal{E}_2^{(2)}\left(\frac{\rho_2}{1-\alpha_1}\right) \right] \\ &= \left(\frac{\rho_1}{\alpha_1}\right)^2 U_{11} + \left(\frac{\rho_2}{1-\alpha_1}\right)^2 U_{22}, \end{aligned}$$

where

$$U_{ii} := \frac{1}{\alpha_i} \left[2\mathcal{E}'_i\left(\frac{\rho_i}{\alpha_i}\right) + \frac{\rho_i}{\alpha_i} \mathcal{E}_i^{(2)}\left(\frac{\rho_i}{\alpha_i}\right) \right] = \frac{\partial^2 U}{\partial \rho_i^2} = \frac{1}{\alpha_i} \frac{d^2 \rho \mathcal{E}_i(\rho)}{d\rho^2} \Big|_{\rho=\rho_i/\alpha_i} > 0,$$

since we assume that $\rho \mathcal{E}_i(\rho)$ is a convex function of ρ for $i = 1, 2$. We are thus allowed to use the implicit functions theorem to express α_1 as a function of ρ_1 and ρ_2 : $\alpha_1 = \alpha_1(\rho_1, \rho_2)$. For $i = 1, 2$, we obtain that

$$\begin{aligned} \nabla \left(\frac{\partial U}{\partial \rho_i} \right) &= \frac{\partial^2 U}{\partial \rho_i \partial \rho_1} \nabla \rho_1 + \frac{\partial^2 U}{\partial \rho_i \partial \rho_2} \nabla \rho_2 + \frac{\partial^2 U}{\partial \rho_i \partial \alpha_1} \nabla \alpha_1 \\ &= \left(\frac{\partial^2 U}{\partial \rho_i \partial \rho_1} + \frac{\partial^2 U}{\partial \rho_i \partial \alpha_1} \frac{\partial \alpha_1}{\partial \rho_1} \right) \nabla \rho_1 + \left(\frac{\partial^2 U}{\partial \rho_i \partial \rho_2} + \frac{\partial^2 U}{\partial \rho_i \partial \alpha_1} \frac{\partial \alpha_1}{\partial \rho_2} \right) \nabla \rho_2 \\ &= \nabla(U_i), \end{aligned}$$

where

$$U_i(\rho_1, \rho_2) := \frac{\partial U}{\partial \rho_i}(\rho_1, \rho_2, \alpha_1(\rho_1, \rho_2))$$

as claimed in Equation (2.11). We also compute that

$$\frac{\partial^2 U}{\partial \rho_1 \partial \alpha_1} = -\frac{1}{\alpha_1} \left[2\frac{\rho_1}{\alpha_1} \mathcal{E}'_1\left(\frac{\rho_1}{\alpha_1}\right) + \left(\frac{\rho_1}{\alpha_1}\right)^2 \mathcal{E}_1^{(2)}\left(\frac{\rho_1}{\alpha_1}\right) \right] = -\frac{\rho_1}{\alpha_1} U_{11},$$

and that

$$\frac{\partial^2 U}{\partial \rho_2 \partial \alpha_1} = \frac{\rho_2}{\alpha_2} U_{22}.$$

Finally, we can use the identity

$$\frac{\partial \alpha_1}{\partial \rho_i} = -\frac{\partial g / \partial \rho_i}{\partial g / \partial \alpha_1} = -\frac{(\partial^2 U / \partial \alpha_1 \partial \rho_i)}{(\partial^2 U / \partial \alpha_1^2)}$$

to obtain that

$$a_i := \frac{\partial U_i}{\partial \rho_i} = \frac{U_{11} U_{22} \left(\frac{\rho_{3-i}}{\alpha_{3-i}} \right)^2}{\left(\frac{\rho_1}{\alpha_1} \right)^2 U_{11} + \left(\frac{\rho_2}{1-\alpha_1} \right)^2 U_{22}} > 0,$$

and

$$\frac{\partial U_1}{\partial \rho_2} = \frac{U_{11} U_{22} \frac{\rho_1}{\alpha_1} \frac{\rho_2}{\alpha_2}}{\left(\frac{\rho_1}{\alpha_1} \right)^2 U_{11} + \left(\frac{\rho_2}{1-\alpha_1} \right)^2 U_{22}} = \frac{\partial U_2}{\partial \rho_1} =: b > 0.$$

It follows that $a_1 a_2 - b^2 = 0$ as claimed.

Remark 2.3. *In the case where one phase is incompressible, there is no dependence on the variable α_1 and the derivation is easier. However, the system obtained still has the same structure, and the equality $a_1 a_2 = b^2$ is obtained by direct computation.*

3. Hyperbolicity of the equations

We give here the proof that System (2.10) is hyperbolic. We first write (2.10) in matricial form:

$$\partial_t Y + \sum_{j=1}^d A_j(Y) \partial_j Y = 0.$$

The vector field Y takes values in $\mathbb{R}^{2+2d+d(d+1)/2} = \mathbb{R}^{(d+1)(d+4)/2}$, and the square matrices $A_j(Y)$ are smooth functions of Y . Let us also define, for every $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$, the matrix

$$A(Y, \xi) := \sum_{j=1}^d A_j(Y) \xi_j.$$

Then saying that (2.10) is hyperbolic is equivalent to saying that, for every $\xi \in \mathbb{R}^d$ such that $|\xi| = 1$, the matrix $A(Y, \xi)$ is diagonalizable with real eigenvalues.

Since the property of hyperbolicity is invariant under C^1 change of variables, we choose to set

$$Y := \begin{pmatrix} \rho_2 \\ v_2 \\ \rho \\ v \\ \tilde{P} \end{pmatrix}, \quad \text{with} \quad \tilde{P} := P > 0 \text{ if } d = 1, \quad \tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{22} \end{pmatrix} \text{ if } d = 2, \quad \tilde{P} := \begin{pmatrix} P_{11} \\ P_{12} \\ P_{13} \\ P_{22} \\ P_{23} \\ P_{33} \end{pmatrix} \text{ if } d = 3$$

(recall that P is a symmetric second order tensor). With this variables, the matrix

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 3. Hyperbolicity of the equations

$A(Y, \xi)$ can be written as a block matrix:

$$A(Y, \xi) = \begin{pmatrix} v_2 \cdot \xi & \rho_2 \xi^T & 0 & 0 & 0 \\ \alpha \xi & (v \cdot \xi) I_d + \xi(v_2 - v)^T & (\beta + P/\rho) \xi & 0 & C(\xi) \\ 0 & 0 & v \cdot \xi & \rho \xi^T & 0 \\ \delta \xi & 0 & (\gamma + P/\rho) \xi & (v \cdot \xi) I_d & C(\xi) \\ 0 & 0 & 0 & D(\xi) & (v \cdot \xi) I_d \end{pmatrix}.$$

The coefficients α, β, γ and δ are given by

$$\alpha := a_2 - b, \quad \beta := b, \quad \gamma := \frac{1}{\rho}(\rho_1 a_1 + \rho_2 b), \quad \delta := \frac{1}{\rho}[\rho_1(b - a_1) + \rho_2(a_2 - b)].$$

When $d = 1$, the matrices $C(\xi)$ and $D(\xi)$ are scalars given by $C(\xi) := \xi$, $D(\xi) := 2P\xi$. When $d = 2$, one has

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & 0 \\ 0 & \xi_1 & \xi_2 \end{pmatrix} \text{ and } D(\xi) := \begin{pmatrix} 2P_{11}\xi_1 + 2P_{12}\xi_2 & 0 \\ P_{21}\xi_1 + P_{22}\xi_2 & P_{11}\xi_1 + P_{12}\xi_2 \\ 0 & 2P_{12}\xi_1 + 2P_{22}\xi_2 \end{pmatrix}.$$

When $d = 3$,

$$C(\xi) := \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ 0 & \xi_1 & 0 & \xi_2 & \xi_3 & 0 \\ 0 & 0 & \xi_1 & 0 & \xi_2 & \xi_3 \end{pmatrix}, \quad D(\xi) := \begin{pmatrix} 2(P\xi)_1 & 0 & 0 \\ (P\xi)_2 & (P\xi)_1 & 0 \\ (P\xi)_3 & 0 & (P\xi)_1 \\ 0 & 2(P\xi)_2 & 0 \\ 0 & (P\xi)_3 & (P\xi)_2 \\ 0 & 0 & 2(P\xi)_3 \end{pmatrix},$$

where $(P\xi)_i$ denotes the i -th component of the vector $P\xi$.

In order to prove the hyperbolicity, the first step is to compute the characteristic polynomial of the matrix $A(Y, \xi)$. This polynomial is given by the following proposition.

Proposition 2.1. *The characteristic polynomial of the matrix $A(Y, \xi)$ is given by*

$$\begin{aligned} \chi_{A(Y, \xi)}(\lambda) = & (v \cdot \xi - \lambda)^{(d+2)(d-1)/2+1} [(v \cdot \xi - \lambda)^2 - \xi^T P \xi]^{d-2} \\ & \times \{2\rho_2 \delta (v \cdot \xi - \lambda)(v_2 - v)^T [P - (\xi^T P \xi)] \xi + [(v \cdot \xi - \lambda)^2 - \xi^T P \xi] q(\lambda)\}, \end{aligned} \quad (2.38)$$

where

$$q(\lambda) := [(v_2 \cdot \xi - \lambda)^2 - \alpha \rho_2] [(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho \gamma] - \rho_2 \delta (3\xi^T P \xi + \rho \beta). \quad (2.39)$$

Let us also denote $A'(Y, \xi)$ the matrix of the simpler system (2.16) (without lift terms). The characteristic polynomial of this system is given by

$$\chi_{A'(Y, \xi)} = (v_2 \cdot \xi - \lambda)^{d-1} (v \cdot \xi - \lambda)^{d(d-1)/2+1} [(v \cdot \xi - \lambda)^2 - \xi^T P \xi]^{d-1} q(\lambda), \quad (2.40)$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 3. Hyperbolicity of the equations

where $q(\lambda)$ is as in Equation (2.39).

The proof of this Proposition is technical and requires a lot of linear algebra computations. Hence we defer it to Appendix 5.1.

Now that we have a formula for the characteristic polynomial of the matrices $A(Y, \xi)$ and $A'(Y, \xi)$, we can study the hyperbolicity of Systems (2.10) and (2.16). We will investigate the cases $d = 1$, $d = 2$ and $d = 3$ separately.

3.1. Case $d = 1$

When $d = 1$, the hyperbolicity is given by the following proposition.

Proposition 2.2. *There exists a constant $C = C(\rho_1, \rho_2, P)$ such that, in dimension 1, System (2.10) and System (2.16) are hyperbolic when $|v_1 - v_2| < C$.*

Proof. When $d = 1$, the second order tensor P is a scalar, as well as the vector ξ . Since we have that $|\xi| = 1$, we deduce that $\xi = \pm 1$ and $P - \xi^T P \xi = 0$. Hence the second line of (2.38) vanishes and the characteristic polynomial is equal to

$$\chi_{A(Y, \xi)} = (v \cdot \xi - \lambda) \left[(v_2 \cdot \xi - \lambda)^2 - \alpha \rho_2 \right] \left[(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho \gamma \right] - \rho_2 \delta (3\xi^T P \xi + \rho \beta).$$

Without loss of generality, we can suppose that $\xi = 1$. We thus obtain that

$$q(\lambda) = [(v_2 - \lambda)^2 - \alpha \rho_2] [(v - \lambda)^2 - 3P - \rho \gamma] - \rho_2 \delta (3P + \rho \beta). \quad (2.41)$$

We need to show that $q(\lambda)$ has four real roots. Since the general study of the roots of this polynomial is complicated, we first restrict ourselves to the case $v_1 = v_2 = v$. If we define $X := (v - \lambda)^2$, then

$$q(\lambda) = (X - \alpha \rho_2)(X - 3P - \rho \gamma) - \rho_2 \delta (3P + \rho \beta) =: Q(X).$$

Hence $q(\lambda) = 0$ if and only if $Q(X) = 0$. Since Q is of degree 2, we have an explicit formula for its roots denoted $r_{\pm}$:

$$r_{\pm} = \frac{\alpha \rho_2 + \rho \gamma + 3P \pm \sqrt{\Delta}}{2},$$

where

$$\begin{aligned} \Delta &:= (\alpha \rho_2 + \rho \gamma + 3P)^2 - 4[(\alpha \rho_2(3P + \rho \gamma) - \rho_2 \delta(3P + \rho \beta))] \\ &= (\alpha \rho_2 - \rho \gamma - 3P)^2 + 4\rho_2 \delta(3P + \rho \beta). \end{aligned} \quad (2.42)$$

We recall that $\alpha, \beta, \gamma, \delta$ are given by

$$\alpha := a_2 - b, \quad \beta := b, \quad \gamma := \frac{1}{\rho}(\rho_1 a_1 + \rho_2 b), \quad \delta := \frac{1}{\rho}[\rho_1(b - a_1) + \rho_2(a_2 - b)].$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 3. Hyperbolicity of the equations

It follows that

$$\alpha\rho_2 + \rho\gamma = \rho_1 a_1 + \rho_2 a_2 > 0, \quad \alpha\rho_2 - \rho\gamma = \rho_2 a_2 - \rho_1 a_1 - 2\rho_2 b = \rho\delta - \rho\beta,$$

hence

$$\begin{aligned} \Delta &= (\rho\delta - \rho\beta - 3P)^2 + 4\rho_2\delta(3P + \rho\beta) \\ &= (\rho\delta + \rho\beta + 3P)^2 - 4\rho_1\delta(3P + \rho\beta). \end{aligned}$$

We can combine the two previous lines to obtain

$$\Delta = \frac{\rho_1}{\rho}(\rho\delta - \rho\beta - 3P)^2 + \frac{\rho_2}{\rho}(\rho\delta + \rho\beta + 3P)^2 > 0,$$

since $\rho\beta + 3P = \rho b + 3P > 0$. Hence the two roots $r_{\pm}$ are real and distinct. If we now return to (2.42), we see that

$$\begin{aligned} \Delta &= (\rho_1 a_1 + \rho_2 a_2 + 3P)^2 - 4[3P\rho_2(\alpha - \delta) + \rho_2\rho(\alpha\gamma - \delta\beta)] \\ &= (\rho_1 a_1 + \rho_2 a_2 + 3P)^2 - 4\left[3P\frac{\rho_1\rho_2}{\rho}(a_2 + a_1 - 2b) + \rho_2\rho_1(a_1 a_2 - b^2)\right] \\ &= (\rho_1 a_1 + \rho_2 a_2 + 3P)^2 - 12P\frac{\rho_1\rho_2}{\rho}(a_2 + a_1 - 2b), \end{aligned}$$

where the last equality comes from the relation (2.13). From the expressions of a_1 , a_2 and b obtained in Section 2, we see that

$$a_1 + a_2 - 2b = \gamma\left(\frac{\rho_1}{\alpha_1} - \frac{\rho_2}{\alpha_2}\right)^2, \quad (2.43)$$

where γ is a positive constant. It follows that

$$r_{\pm} = \frac{\rho_1 a_1 + \rho_2 a_2 + 3P \pm \sqrt{(\rho_1 a_1 + \rho_2 a_2 + 3P)^2 - 12P\frac{\rho_1\rho_2}{\rho}\gamma\left(\frac{\rho_1}{\alpha_1} - \frac{\rho_2}{\alpha_2}\right)^2}}{2}.$$

We see that $r_+ > 0$ and $r_- \geq 0$, with $r_- = 0$ if and only if $\rho_1/\alpha_1 = \rho_2/\alpha_2$, i.e. if and only if the two material densities are equal. Since by hypothesis the two material densities are different, we also have $r_- > 0$. Hence $\chi_{A(Y,\xi)}$ has five distinct roots, given by

$$\lambda_1 = v, \quad \lambda_{2\pm} = v \pm \sqrt{r_+}, \quad \lambda_{3\pm} = v \pm \sqrt{r_-}.$$

By continuity, there exists $C = C(\rho_1, \rho_2, P)$ such that $\chi_{A(Y,\xi)}$ has five distinct roots for every v_1, v_2 such that $|v_1 - v_2| \leq C(\rho_1, \rho_2, P)$. Hence the matrix $A(Y, \xi)$ is diagonalizable with real eigenvalues and System (2.10) is strictly hyperbolic. The proof for the system without lift terms is the same since these terms vanish when $d = 1$. □

3.2. Case $d = 2$

We now move on to the case $d = 2$. Recall that we aim to prove the following result:

Proposition 2.3. *Let θ be defined as in Equation (2.17) of Theorem (2.1):*

$$\theta := \rho_1 \rho_2 (\sqrt{a_1} - \sqrt{a_2})^2 - \frac{1}{2} (\rho_1 \sqrt{a_1} + \rho_2 \sqrt{a_2})^2.$$

If $\rho P < \theta I_d$ or $\rho P > \theta I_d$, there exists $C = C(\rho_1, \rho_2, P) > 0$ such that Systems (2.10) and (2.16) are hyperbolic, whenever $|v_1 - v_2| < C$.

Proof. In dimension 2, the characteristic polynomial of $A(Y, \xi)$ is given by

$$\chi_{A(Y, \xi)} = (v \cdot \xi - \lambda)^3 \left[((v \cdot \xi - \lambda)^2 - \xi^T P \xi) q(\lambda) + 2\rho_2 \delta(v \cdot \xi - \lambda)(v_2 - v)^T [P - (\xi^T P \xi)] \xi \right],$$

where $q(\lambda)$ was defined in Equation (2.39). Let us first look at the polynomial

$$\tilde{q}(\lambda) := [(v \cdot \xi - \lambda)^2 - \xi^T P \xi] q(\lambda) + 2\rho_2 \delta(v \cdot \xi - \lambda)(v_2 - v)^T [P - (\xi^T P \xi)] \xi,$$

such that

$$\chi_{A(Y, \xi)} = (v \cdot \xi - \lambda)^3 \tilde{q}(\lambda).$$

When $v_1 = v_2 = v$, $\tilde{q}$ reduces to

$$\tilde{q}(\lambda) = [(v \cdot \xi - \lambda)^2 - \xi^T P \xi] q(\lambda).$$

By mimicking the case $d = 1$, we deduce that $\tilde{q}$ has six roots, given by

$$\lambda_{1\pm} = v \cdot \xi \pm \sqrt{\xi^T P \xi}, \quad \lambda_{2\pm} = v \cdot \xi \pm \sqrt{r_+}, \quad \lambda_{3\pm} = v \cdot \xi \pm \sqrt{r_-},$$

with

$$r_{\pm} = \frac{\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P \xi \pm \sqrt{(\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P \xi)^2 - 12\xi^T P \xi \frac{\rho_1 \rho_2}{\rho} \gamma \left(\frac{\rho_1}{a_1} - \frac{\rho_2}{a_2} \right)^2}}{2}. \quad (2.44)$$

Let us show that the six roots are all distinct. From the case $d = 1$, we see that $\lambda_{2\pm}$ and $\lambda_{3\pm}$ are always distinct. We also see with Equation (2.44) that $r_+ > \xi^T P \xi$, hence $\lambda_{1\pm}$ and $\lambda_{2\pm}$ are also always different. If finally we suppose that $\lambda_{1\pm}$ and $\lambda_{3\pm}$ are not distinct, we obtain from Equations (2.44) and (2.43) the following equality:

$$\rho \xi^T P \xi = \rho_1 \rho_2 (a_1 + a_2 - 2b) - \frac{1}{2} (\rho_1^2 a_1 + \rho_2^2 a_2 + 2\rho_1 \rho_2 b),$$

which yields after replacing b by $\sqrt{a_1 a_2}$,

$$\rho \xi^T P \xi = \rho_1 \rho_2 (\sqrt{a_1} - \sqrt{a_2})^2 - \frac{1}{2} (\rho_1 \sqrt{a_1} + \rho_2 \sqrt{a_2})^2 = \theta,$$

where θ was defined in (2.17). Since we supposed that $\rho P > \theta I_d$ or that $\rho P < \theta I_d$, and $\xi^T \xi = 1$, we deduce that $\rho \xi^T P \xi \neq \theta$ and the six roots are real and distinct.

By continuity, there exists $C = C(\rho_1, \rho_2, P)$ such that $\tilde{q}(\lambda)$ has six distinct real roots for every v_1, v_2 such that $|v_1 - v_2| \leq C(\rho_1, \rho_2, P)$. Up to reducing C , we can suppose that all six eigenvalues are different from $v \cdot \xi$. Since the six eigenvalues are distinct, there exist six corresponding eigenvectors. We are thus left to find three independent eigenvectors for the eigenvalue $\lambda = v \cdot \xi$. The matrix $A(Y, \xi) - (v \cdot \xi) I_d$ is given by

$$A(Y, \xi) - (v \cdot \xi) I_d = \begin{pmatrix} v_2 \cdot \xi - v \cdot \xi & \rho_2 \xi^T & 0 & 0 & 0 \\ \alpha \xi & \xi(v_2 - v)^T & (\beta + P/\rho) \xi & 0 & C(\xi) \\ 0 & 0 & 0 & \rho \xi^T & 0 \\ \delta \xi & 0 & (\gamma + P/\rho) \xi & 0 & C(\xi) \\ 0 & 0 & 0 & D(\xi) & 0 \end{pmatrix}$$

Let X be an eigenvector of $A(Y, \xi)$ for the eigenvalue $(v \cdot \xi)$. We write X as a block vector compatible with $A(Y, \xi)$ and solve the linear system $[A(Y, \xi) - (v \cdot \xi) I_d] X = 0$. After some computations, we obtain the following three independent eigenvectors:

$$X_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ \xi^\perp \\ \frac{(v_2 - v) \cdot \xi^\perp}{\gamma - \beta} \\ 0 \\ u_2 \end{pmatrix}, \quad X_3 = \begin{pmatrix} \rho_2 \\ (v \cdot \xi - v_2 \cdot \xi) \xi \\ \frac{\rho_2 \delta - \rho_2 \alpha + [(v_2 - v) \cdot \xi]^2}{\beta - \gamma} \\ 0 \\ u_3 \end{pmatrix}, \quad (2.45)$$

where $\xi^\perp, u_1, u_2, u_3 \in \mathbb{R}^2$ are four vectors such that

$$\xi^\perp \neq 0, \quad \xi \cdot \xi^\perp = 0, \quad u_1 \neq 0, \quad C(\xi) u_1 = 0, \quad C(\xi) u_2 = -\frac{(v_2 - v) \cdot \xi^\perp}{\gamma - \beta} \left(\gamma + \frac{P}{\rho} \right) \xi,$$

and

$$C(\xi) u_3 = \left[-\rho_2 \delta - \frac{\rho_2 \delta - \rho_2 \alpha + [(v_2 - v) \cdot \xi]^2}{\beta - \gamma} \left(\gamma + \frac{P}{\rho} \right) \right] \xi.$$

In order to see that the three vectors are well defined, we first note that $\gamma - \beta = \rho_1(b - a_1)/\rho \neq 0$, since otherwise $b = a_1$ and from $a_1 a_2 = b^2$ one obtains $a_1 = a_2 = b$, which is incompatible with the assumption that $a_1 + a_2 - 2b > 0$. Furthermore, we see that the matrix $C(\xi)$ has always a rank 2 submatrix when $\xi \neq 0$, which implies that $C(\xi)$ is onto and the dimension of its kernel is one. Thus the vectors u_1, u_2, u_3 are well defined. Hence the matrix $A(Y, \xi)$ is diagonalizable and System (2.10) is only weakly hyperbolic.

Case of the system without lift terms

Finally, let us investigate the case where the lift terms vanish. In this case, the characteristic polynomial is given by

$$\chi_{A'(Y,\xi)}(\lambda) = (v_2 \cdot \xi - \lambda)(v \cdot \xi - \lambda)^2[(v \cdot \xi - \lambda)^2 - \xi^T P \xi]q(\lambda).$$

When $v_1 = v_2 = v$, it follows from the case $d = 1$ that $q(\lambda)$ has four distinct real roots, given by

$$\lambda_{1\pm} = v \cdot \xi \pm \sqrt{r_+}, \quad \lambda_{2\pm} = v \cdot \xi \pm \sqrt{r_-}.$$

Hence there exists a constant $C = C(\rho_1, \rho_2, P)$ such that q has four distinct real roots, for any $|v_1 - v_2| < C(\rho_1, \rho_2, P)$. This implies that $\chi_{A'(Y,\xi)}$ also only has real roots for these values of v_1, v_2 . As in the previous case, up to reducing C , we can assume that the roots of q are distinct from $v \cdot \xi$, from $v_2 \cdot \xi$ and from $v \cdot \xi \pm \sqrt{\xi^T P \xi}$ by using the condition $\rho P < \theta I_d$ or $\rho P > \theta I_d$. It follows that most of the eigenvalues have a multiplicity one, except for $v \cdot \xi$, which has multiplicity two (or three if $v_2 \cdot \xi = v \cdot \xi$). Thus we need to find two eigenvectors for this eigenvalue $v \cdot \xi$, and three when $v_2 \cdot \xi = v \cdot \xi$. We compute that

$$A'(Y, \xi) - (v \cdot \xi)I_d = \begin{pmatrix} v_2 \cdot \xi - v \cdot \xi & \rho_2 \xi^T & 0 & 0 & 0 \\ \alpha \xi & (v_2 \cdot \xi - v \cdot \xi)I_d & (\beta + P/\rho)\xi & 0 & C(\xi) \\ 0 & 0 & 0 & \rho \xi^T & 0 \\ \delta \xi & 0 & (\gamma + P/\rho)\xi & 0 & C(\xi) \\ 0 & 0 & 0 & D(\xi) & 0 \end{pmatrix}$$

We find two independent eigenvectors, given by

$$X_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_1 \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} \rho_2 \\ (v_2 \cdot \xi - v \cdot \xi)\xi \\ \frac{\rho_2(\alpha - \delta) + (v_2 \cdot \xi - v \cdot \xi)^2}{\gamma - \beta} \\ 0 \\ u_3 \end{pmatrix}, \quad (2.46)$$

where $u_1 \neq 0$, $C(\xi)u_1 = 0$ and

$$C(\xi)u_3 = \left[-\rho_2 \delta - \frac{\rho_2 \delta - \rho_2 \alpha + [(v_2 - v) \cdot \xi]^2}{\beta - \gamma} \left(\gamma + \frac{P}{\rho} \right) \right] \xi.$$

Note that, when $v_2 \cdot \xi = v \cdot \xi$, a third eigenvector for the eigenvalue $v \cdot \xi$ is

$$X_2 = \begin{pmatrix} 0 \\ \xi^\perp \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{with} \quad \xi^\perp \neq 0 \quad \text{and} \quad \xi^T \xi^\perp = 0. \quad (2.47)$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 3. Hyperbolicity of the equations

Finally, we can show that the condition $\rho P < \theta I_d$ or $\rho P > \theta I_d$ is necessary for the hyperbolicity of System (2.16). Indeed, if this condition does not hold, one can find a vector ξ such that $|\xi| = 1$ and that $q(v \cdot \xi \pm \sqrt{\xi^T P \xi}) = 0$, whenever v_1 and v_2 are close enough. In this case, $v \cdot \xi \pm \sqrt{\xi^T P \xi}$ is a root of $\chi_{A'(Y, \xi)}$ of multiplicity two. However, computations show that the kernel of the matrix $A'(Y, \xi) - (v \cdot \xi \pm \sqrt{\xi^T P \xi}) I_d$ is always of dimension 1. Hence the matrix $A'(Y, \xi)$ is not diagonalisable in this case and System (2.16) is not hyperbolic. $\square$

3.3. Case $d = 3$

In dimension 3, we need to prove the following proposition:

Proposition 2.4. *Let θ be defined as in Equation (2.17) of Theorem (2.1):*

$$\theta := \rho_1 \rho_2 (\sqrt{a_1} - \sqrt{a_2})^2 - \frac{1}{2} (\rho_1 \sqrt{a_1} + \rho_2 \sqrt{a_2})^2.$$

If $\rho P < \theta I_d$ or $\rho P > \theta I_d$, there exists $C = C(\rho_1, \rho_2, P) > 0$ such that System (2.16) is hyperbolic, whenever $|v_1 - v_2| < C$. Under these assumptions, System (2.10) is weakly hyperbolic.

Proof. We first prove the result concerning System (2.10). When $d = 3$, the characteristic polynomial of the system is given by

$$\chi_{A(Y, \xi)} = (v \cdot \xi - \lambda)^6 [(v \cdot \xi - \lambda)^2 - \xi^T P \xi] \tilde{q}(\lambda),$$

with

$$\tilde{q}(\lambda) = [(v \cdot \xi - \lambda)^2 - \xi^T P \xi] q(\lambda) + 2\rho_2 \delta(v \cdot \xi - \lambda)(v_2 - v)^T [P - (\xi^T P \xi)] \xi$$

and $q(\lambda)$ was defined in Equation (2.39). When ξ is such that

$$(v_2 - v)^T [P - (\xi^T P \xi)] \xi = 0, \tag{2.48}$$

the characteristic polynomial can be written

$$\chi_{A(Y, \xi)} = (v \cdot \xi - \lambda)^6 [(v \cdot \xi - \lambda)^2 - \xi^T P \xi]^2 q(\lambda).$$

Hence the algebraic multiplicity of $v \cdot \xi \pm \sqrt{\xi^T P \xi}$ is equal to two. However, computations show that the geometric multiplicity of this eigenvalue is equal to two if and only if the vectors $v_2 - v$ and ξ are proportional, and is equal to one when it is not the case. Since Equation (2.48) can be satisfied even when ξ and $v_2 - v$ are not proportional (for instance when ξ is an eigenvector of P), the matrix $A(Y, \xi)$ is not diagonalizable for every ξ and System (2.10) is not hyperbolic in dimension 3.

Case of the system without lift terms

Let us now consider System (2.16). When $d = 3$,

$$\chi_{A'(Y,\xi)} = (v_2 \cdot \xi - \lambda)^2 (v \cdot \xi - \lambda)^4 [(v \cdot \xi - \lambda)^2 - \xi^T P \xi]^2 q(\lambda).$$

From the case $d = 1$, there exists $C = C(\rho_1, \rho_2, P) > 0$ such that q has four distinct real roots whenever $|v_1 - v_2| < C$. Up to reducing C , we can suppose that the four roots of q are distinct from $v \cdot \xi$, from $v_2 \cdot \xi$ and from $v \cdot \xi \pm \sqrt{\xi^T P \xi}$.

We are thus left to find two independent eigenvectors for the eigenvalue $v_2 \cdot \xi$, four for the eigenvalue $v \cdot \xi$ and two for the eigenvalues $v \cdot \xi \pm \sqrt{\xi^T P \xi}$. First, we see that the matrix $C(\xi)$ is of rank 3 since it always has an invertible 3×3 submatrix. Thus the dimension of its kernel is also 3. Hence the formulas for the vectors X_1 and X_3 , that were defined in Equation (2.46) for the case $d = 2$ already provide 4 independent eigenvectors. Similarly, the subspace which is orthogonal to ξ is of dimension 2 when $d = 3$, hence the Equation (2.47) defines two independent eigenvectors for the eigenvalue $v_2 \cdot \xi$.

Finally, the matrix $A'(Y, \xi) - (v \cdot \xi \mp \sqrt{\xi^T P \xi}) I_d$ is equal to

$$\begin{pmatrix} v_2 \cdot \xi - v \cdot \xi \pm \sqrt{\xi^T P \xi} & \rho_2 \xi^T & 0 & 0 & 0 \\ \alpha \xi & (v_2 \cdot \xi - v \cdot \xi \pm \sqrt{\xi^T P \xi}) I_d & (\beta + P/\rho) \xi & 0 & C(\xi) \\ 0 & 0 & \pm \sqrt{\xi^T P \xi} & \rho \xi^T & 0 \\ \delta \xi & 0 & (\gamma + P/\rho) \xi & \pm \sqrt{\xi^T P \xi} I_d & C(\xi) \\ 0 & 0 & 0 & D(\xi) & \pm \sqrt{\xi^T P \xi} I_d \end{pmatrix}.$$

We obtain two eigenvectors of the form

$$X_4 = \begin{pmatrix} 0 \\ \pm \sqrt{\xi^T P \xi} \xi^\perp \\ 0 \\ (v_2 \cdot \xi - v \cdot \xi \pm \sqrt{\xi^T P \xi}) \xi^\perp \\ \mp \frac{v_2 \cdot \xi - v \cdot \xi \pm \sqrt{\xi^T P \xi}}{\sqrt{\xi^T P \xi}} D(\xi) \xi^\perp \end{pmatrix}, \quad \text{where } \xi^\perp \neq 0 \quad \text{and} \quad \xi^T \xi^\perp = 0.$$

Hence $A'(Y, \xi)$ is diagonalizable and System (2.16) is hyperbolic. □

4. Other models

This section is devoted to the proofs of Theorems 2.2, 2.3 and 2.4. The proofs of these theorems is very similar to the proof of Theorem 2.1. As a consequence, the proofs are given without detailed computations.

4.1. Case where the turbulence is carried by one phase

In this part, we give the proof of Theorem 2.2. We thus consider a two-phase flow for which one of the two phases, say the phase 2, is turbulent. In other words, instead of assuming that the turbulence is created by the mixture of the two phases, as it was the case in System (2.10), we suppose that it is intrinsic to the phase 2. We choose for this two-phase flow the Lagrangian defined in Equation (2.19):

$$\mathcal{L} := \frac{1}{2}\rho_1|v_1|^2 + \frac{1}{2}\rho_2|v_2|^2 - \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) - \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) - \frac{1}{2}\rho_2\text{Tr}(P_2). \quad (2.49)$$

We use the superscript 2 for the tensor P_2 to insist on the fact that it is carried by the phase two. as a consequence, we assume that the evolution of P_2 is given by Equation (2.20):

$$\partial_t P_2 + (v_2 \cdot \nabla)P_2 + \frac{\partial v_2}{\partial x}P_2 + P_2\left(\frac{\partial v_2}{\partial x}\right)^T = 0. \quad (2.50)$$

We impose again the conservation of the two densities:

$$\partial_t \rho_i + \text{div}(\rho_i v_i) = 0, \quad i = 1, 2. \quad (2.51)$$

We also define, for $i = 1, 2$, the trajectory followed by the particles of the phase i , denoted x_i , such that

$$\partial_t x_i(t, X) = v_i(t, x_i(t, X)), \quad x_i(0, X) = X.$$

The action associated to the Lagrangian (2.49) is given by

$$\mathcal{A}[x_1, x_2, \alpha_1^l] := \int_0^T \int_{\Omega} \mathcal{L}.$$

One can apply the stationary action principle under the constraints given by (2.51) and (2.50) to obtain the following system:

$$\left\{ \begin{array}{ll} \partial_t \rho_i + \text{div}(\rho_i v_i) = 0, & i = 1, 2, \quad (2.52a) \\ \partial_t v_1 + (v_1 \cdot \nabla)v_1 + \nabla(U_1) = 0, & (2.52b) \\ \partial_t v_2 + (v_2 \cdot \nabla)v_2 + \nabla(U_2) + \frac{1}{\rho_2}\text{div}(\rho_2 P_2)^T = 0, & (2.52c) \\ \partial_t P_2 + (v_2 \cdot \nabla)P_2 + \frac{\partial v_2}{\partial x}P_2 + P_2\left(\frac{\partial v_2}{\partial x}\right)^T = 0. & (2.52d) \end{array} \right.$$

The potentials U_1 and U_2 are defined as in Section 1. The conservation of total momentum still holds:

$$\partial_t(\rho_1 v_1 + \rho_2 v_2) + \text{div}(\rho_1 v_1 \otimes v_1 + \rho_2 v_2 \otimes v_2 + pI_d + \rho_2 P_2)^T = 0,$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 4. Other models

as well as the conservation of energy:

$$\partial_t(e_1 + e_2) + \operatorname{div}(e_1 v_1 + e_2 v_2 + p(\alpha_1 v_1 + \alpha_2 v_2) + \rho_2 P_2 v_2) = 0,$$

where

$$e_1 := \frac{1}{2} \rho_1 |v_1|^2 + \rho_1 \mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right), \quad e_2 := \frac{1}{2} \rho_2 |v_2|^2 + \rho_2 \mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) + \frac{1}{2} \rho_2 \operatorname{Tr}(P_2)$$

and

$$p := \left(\frac{\rho_1}{\alpha_1}\right)^2 \mathcal{E}'_1\left(\frac{\rho_1}{\alpha_1}\right) = \left(\frac{\rho_2}{\alpha_2}\right)^2 \mathcal{E}'_2\left(\frac{\rho_2}{\alpha_2}\right).$$

We now recall the main result about System (2.52):

Proposition 2.5. *The hyperbolicity of System (2.52) can be characterized as follows:*

- When $d = 1$, System (2.52) is hyperbolic for small relative velocity $|v_1 - v_2|$, i.e. there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that (2.52) is hyperbolic when $|v_1 - v_2| \leq C$.
- When $d = 2, 3$ let

$$\mu := \rho_1 a_1 - \frac{1}{2} \rho_2 a_2$$

and assume that either $P_2 > \mu I_d$ or $P_2 < \mu I_d$. Then (2.52) is hyperbolic whenever $|v_1 - v_2| \leq C$.

Proof. We use here the notations of Section 3. The matrix of System (2.52) is given by

$$B(Y, \xi) := \begin{pmatrix} v_1 \cdot \xi & \rho_1 \xi^T & 0 & 0 & 0 \\ a_1 \xi & (v_1 \cdot \xi) I_d & b \xi & 0 & 0 \\ 0 & 0 & v_2 \cdot \xi & \rho_2 \xi^T & 0 \\ b \xi & 0 & (a_2 + \frac{P_2}{\rho_2}) \xi & (v_2 \cdot \xi) I_d & C(\xi) \\ 0 & 0 & 0 & D(\xi) & (v_2 \cdot \xi) I_d \end{pmatrix}$$

We compute that the characteristic polynomial $\chi_{B(Y, \xi)}$ of this matrix is given by

$$\chi_{B(Y, \xi)}(\lambda) = (v_1 \cdot \xi - \lambda)^{d-1} (v_2 \cdot \xi - \lambda)^{d(d-1)/2+1} [(v_2 \cdot \xi - \lambda)^2 - \xi^T P_2 \xi]^{d-2} q_0(\lambda),$$

where

$$q_0(\lambda) := [(v_1 \cdot \xi - \lambda)^2 - \rho_1 a_1] [(v_2 \cdot \xi - \lambda)^2 - \rho_2 a_2 - 3 \xi^T P_2 \xi] - \rho_1 \rho_2 b^2.$$

Let us show that q_0 has four distinct real roots when v_1 and v_2 are close. When $v_1 = v_2 = v$, let us denote $X := (v \cdot \xi - \lambda)^2$. Then

$$q_0(\lambda) = X^2 - (\rho_1 a_1 + \rho_2 a_2 + 3 \xi^T P_2 \xi) X + 3 \xi^T P_2 \xi \rho_1 a_1 =: Q_0(X).$$

Hence $q_0(\lambda) = 0$ if and only if $Q_0(X) = 0$. The roots of Q_0 are given by

$$r_{\pm} = \frac{\rho_1 a_1 + \rho_2 a_2 + 3 \xi^T P_2 \xi \pm \sqrt{\Delta}}{2},$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 4. Other models

where

$$\begin{aligned}\Delta &:= (\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P_2 \xi)^2 - 12\xi^T P_2 \xi \rho_1 a_1 \\ &= (\rho_1 a_1 - \rho_2 a_2 - 3\xi^T P_2 \xi)^2 + 4\rho_1 \rho_2 a_1 a_2 > 0.\end{aligned}$$

Hence $r_{\pm} > 0$ and q_0 has four distinct real roots, given by

$$\lambda_{1\pm} := v \cdot \xi \pm \sqrt{r_+}, \quad \lambda_{2\pm} := v \cdot \xi \pm \sqrt{r_-}.$$

By continuity, there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that q_0 has four distinct real roots, for any $|v_1 - v_2| < C$. For these values of $|v_1 - v_2|$, $\chi_{B(Y, \xi)}$ has only real roots. Since $r_{\pm} > 0$, the roots of q_0 are distinct from $v \cdot \xi$. The assumption $P_2 < \mu I_d$ or $P_2 > \mu I_d$ ensures that the roots of q_0 are also distinct from $v \cdot \xi \pm \sqrt{\xi^T P_2 \xi}$. Thus, up to reducing C , we can always assume that the roots of q_0 are roots of $\chi_{B(Y, \xi)}$ of multiplicity one. It follows that $v_1 \cdot \xi$ is a root of multiplicity $d - 1$, $v_2 \cdot \xi$ is of multiplicity $d(d - 1)/2 + 1$ and $v_2 \cdot \xi \pm \sqrt{\xi^T P_2 \xi}$ is of multiplicity $d - 1$.

Let us show that $B(Y, \xi)$ is diagonalizable. We see that we can find $d - 1$ eigenvectors for the eigenvalue $v_1 \cdot \xi$ which are of the form

$$X_1 := \begin{pmatrix} 0 \\ \xi^\perp \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where } \xi^\perp \neq 0 \quad \text{and} \quad \xi^T \xi^\perp = 0.$$

For the eigenvalue $v_2 \cdot \xi$, we can find $d(d - 1)/2 + 1$ eigenvectors of the form

$$X_2 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u_1 \end{pmatrix} \quad \text{and} \quad X_3 := \begin{pmatrix} \rho_1 \\ (v_2 \cdot \xi - v_1 \cdot \xi) \xi \\ \frac{(v_1 \cdot \xi - v_2 \cdot \xi)^2 - \rho_1 a_1}{b} \\ 0 \\ u_2 \end{pmatrix},$$

where $u_1 \neq 0$, $C(\xi)u_1 = 0$, and

$$C(\xi)u_2 = \left[\frac{\rho_1 a_1 - (v_1 \cdot \xi - v_2 \cdot \xi)^2}{b} \left(a_2 + \frac{P_2}{\rho_2} \right) - \rho_1 b \right] \xi.$$

Note that $C(\xi)$ is a matrix of size $d(d + 1)/2 \times d$ and of rank d , hence the dimension of its kernel is $d(d - 1)/2$.

Finally, we can find $d - 1$ eigenvectors corresponding to the eigenvalue $v_2 \cdot \xi \pm$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 4. Other models

$\sqrt{\xi^T P_2 \xi}$, given by

$$X_4 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi^\perp \\ \pm \frac{1}{\sqrt{\xi^T P_2 \xi}} D(\xi) \xi^\perp \end{pmatrix}, \quad \text{where } \xi^\perp \neq 0 \quad \text{and} \quad \xi^T \xi^\perp = 0.$$

Hence $B(Y, \xi)$ is diagonalizable and System (2.52) is hyperbolic. □

4.2. Case where each phase is turbulent

We now give the proof of Theorem 2.2. We thus consider a two-phase flow in which each phase is endowed with a Reynolds stress tensor denoted P_i , $i = 1, 2$. The Lagrangian density is the following:

$$\mathcal{L} := \frac{1}{2} \rho_1 |v_1|^2 + \frac{1}{2} \rho_2 |v_2|^2 - \rho_1 \mathcal{E}_1 \left(\frac{\rho_1}{\alpha_1} \right) - \rho_2 \mathcal{E}_2 \left(\frac{\rho_2}{\alpha_2} \right) - \frac{1}{2} \rho_1 \text{Tr}(P_1) - \frac{1}{2} \rho_2 \text{Tr}(P_2). \quad (2.53)$$

The densities are still conserved:

$$\partial_t \rho_i + \text{div}(\rho_i v_i) = 0, \quad i = 1, 2.$$

The evolution equations for P_1, P_2 are given by

$$\partial_t P_i + (v_i \cdot \nabla) P_i + \frac{\partial v_i}{\partial x} P_i + P_i \left(\frac{\partial v_i}{\partial x} \right)^T = 0, \quad i = 1, 2.$$

The action associated to the Lagrangian density (2.22) is

$$\mathcal{A} [x_1, x_2, \alpha_1^l] := \int_0^T \int_\Omega \mathcal{L},$$

where the trajectories x_i are defined by

$$\partial_t x_i(t, X) = v_i(t, x_i(t, X)), \quad x_i(0, X) = X.$$

The stationary action principle then gives the following system of equations:

$$\left\{ \begin{array}{ll} \partial_t \rho_i + \text{div}(\rho_i v_i) = 0, & i = 1, 2, \quad (2.54a) \\ \partial_t v_i + (v_i \cdot \nabla) v_i + \nabla(U_i) + \frac{1}{\rho_i} \text{div}(\rho_i P_i)^T = 0, & i = 1, 2 \quad (2.54b) \\ \partial_t P_i + (v_i \cdot \nabla) P_i + \frac{\partial v_i}{\partial x} P_i + P_i \left(\frac{\partial v_i}{\partial x} \right)^T = 0 & i = 1, 2. \quad (2.54c) \end{array} \right.$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 4. Other models

The potentials U_1 and U_2 are defined as in Section 1. The conservation of total momentum still holds:

$$\partial_t(\rho_1 v_1 + \rho_2 v_2) + \operatorname{div}(\rho_1 v_1 \otimes v_1 + \rho_2 v_2 \otimes v_2 + pI_d + \rho_1 P_1 + \rho_2 P_2)^T = 0,$$

as well as the conservation of energy:

$$\partial_t(e_1 + e_2) + \operatorname{div}(e_1 v_1 + e_2 v_2 + p(\alpha_1 v_1 + \alpha_2 v_2) + \rho_1 P_1 v_1 + \rho_2 P_2 v_2) = 0,$$

where

$$e_1 := \frac{1}{2}\rho_1|v_1|^2 + \rho_1\mathcal{E}_1\left(\frac{\rho_1}{\alpha_1}\right) + \frac{1}{2}\rho_1\operatorname{Tr}(P_1), \quad e_2 := \frac{1}{2}\rho_2|v_2|^2 + \rho_2\mathcal{E}_2\left(\frac{\rho_2}{\alpha_2}\right) + \frac{1}{2}\rho_2\operatorname{Tr}(P_2)$$

and

$$p := \left(\frac{\rho_1}{\alpha_1}\right)^2 \mathcal{E}'_1\left(\frac{\rho_1}{\alpha_1}\right) = \left(\frac{\rho_2}{\alpha_2}\right)^2 \mathcal{E}'_2\left(\frac{\rho_2}{\alpha_2}\right).$$

We now state the main result about System (2.54):

Proposition 2.6. *The hyperbolicity of System (2.54) can be characterized as follows:*

- When $d = 1$, System (2.54) is hyperbolic for small relative velocity $|v_1 - v_2|$, i.e. there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that (2.52) is hyperbolic when $|v_1 - v_2| \leq C$.
- When $d = 2, 3$ assume that

$$\frac{1}{3}P_1 \leq P_2 \leq 3P_1. \quad (2.55)$$

Then (2.52) is hyperbolic whenever $|v_1 - v_2| \leq C$.

Proof. We use here the notations of Section 3. The matrix of System (2.52) is given by

$$C(Y, \xi) := \begin{pmatrix} v_1 \cdot \xi & \rho_1 \xi^T & 0 & 0 & 0 & 0 \\ (a_1 + \frac{P_1}{\rho_1})\xi & (v_1 \cdot \xi)I_d & C(\xi) & b\xi & 0 & 0 \\ 0 & D_1(\xi) & (v_1 \cdot \xi)I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & v_2 \cdot \xi & \rho_2 \xi^T & 0 \\ b\xi & 0 & 0 & (a_2 + \frac{P_2}{\rho_2})\xi & (v_2 \cdot \xi)I_d & C(\xi) \\ 0 & 0 & 0 & 0 & D_2(\xi) & (v_2 \cdot \xi)I_d \end{pmatrix},$$

where

$$Y := \begin{pmatrix} \rho_1 \\ v_1 \\ \tilde{P}_1 \\ \rho_2 \\ v_2 \\ \tilde{P}_2 \end{pmatrix}$$

and $D_i(\xi)$ is defined as in Section 3, but with P_i instead of P ($i = 1, 2$). We compute

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 4. Other models

that the characteristic polynomial $\chi_{C(Y,\xi)}$ of this matrix is given by

$$\begin{aligned}\chi_{C(Y,\xi)}(\lambda) &= (\nu_1 \cdot \xi - \lambda)^{d(d-1)/2+1} [(\nu_1 \cdot \xi - \lambda)^2 - \xi^T P_1 \xi]^{d-2} \\ &\quad \times (\nu_2 \cdot \xi - \lambda)^{d(d-1)/2+1} [(\nu_2 \cdot \xi - \lambda)^2 - \xi^T P_2 \xi]^{d-2} q_1(\lambda),\end{aligned}$$

where

$$q_1(\lambda) := [(\nu_1 \cdot \xi - \lambda)^2 - \rho_1 a_1 - 3\xi^T P_1 \xi] [(\nu_2 \cdot \xi - \lambda)^2 - \rho_2 a_2 - 3\xi^T P_2 \xi] - \rho_1 \rho_2 b^2.$$

Let us show that q_1 has four distinct real roots when ν_1 and ν_2 are close. When $\nu_1 = \nu_2 = \nu$, let us denote $X := (\nu \cdot \xi - \lambda)^2$. Then

$$\begin{aligned}q_1(\lambda) &= X^2 - (\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P_1 \xi + 3\xi^T P_2 \xi) X \\ &\quad + (\rho_1 a_1 + 3\xi^T P_1 \xi)(\rho_2 a_2 + 3\xi^T P_2 \xi) - \rho_1 \rho_2 a_1 a_2 \\ &=: Q_1(X).\end{aligned}$$

Hence $q_1(\lambda) = 0$ if and only if $Q_1(X) = 0$. The roots of Q_1 are given by

$$r_{\pm} = \frac{\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P_1 \xi + 3\xi^T P_2 \xi \pm \sqrt{\Delta}}{2},$$

where

$$\begin{aligned}\Delta &:= (\rho_1 a_1 + \rho_2 a_2 + 3\xi^T P_1 \xi + 3\xi^T P_2 \xi)^2 - 4[(\rho_1 a_1 + 3\xi^T P_1 \xi)(\rho_2 a_2 + 3\xi^T P_2 \xi) - \rho_1 \rho_2 a_1 a_2] \\ &= (\rho_1 a_1 + 3\xi^T P_1 \xi - \rho_2 a_2 - 3\xi^T P_2 \xi)^2 + 4\rho_1 \rho_2 a_1 a_2 > 0.\end{aligned}$$

Hence $r_{\pm} > 0$ and q_1 has four distinct real roots, given by

$$\lambda_{1\pm} := \nu \cdot \xi \pm \sqrt{r_+}, \quad \lambda_{2\pm} := \nu \cdot \xi \pm \sqrt{r_-}.$$

By continuity, there exists $C = C(\rho_1, \rho_2, P_2) > 0$ such that q_1 has four distinct real roots, for any $|\nu_1 - \nu_2| < C$. For these values of $|\nu_1 - \nu_2|$, $\chi_{C(Y,\xi)}$ has only real roots. Since $r_{\pm} > 0$, the roots of q_1 are distinct from $\nu \cdot \xi$. The assumption (2.24) ensures that the roots of q_1 are also distinct from $\nu \cdot \xi \pm \sqrt{\xi^T P_i \xi}$ for $i = 1, 2$. Thus, up to reducing C , we can always assume that the roots of q_1 are roots of $\chi_{C(Y,\xi)}$ of multiplicity one. It follows that $\nu_i \cdot \xi$ is a root of multiplicity $d(d-1)/2 + 1$ and $\nu_i \cdot \xi \pm \sqrt{\xi^T P_i \xi}$ is of multiplicity $d-1$, for $i = 1, 2$.

Let us show that $C(Y, \xi)$ is diagonalizable. We see that we can find $d(d-1)/2 + 1$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 4. Other models

eigenvectors for the eigenvalue $v_1 \cdot \xi$ which are of the form

$$X_1 := \begin{pmatrix} 0 \\ 0 \\ u_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad X_2 := \begin{pmatrix} \rho_2 a_2 + 3\xi^T P_2 \xi - (v_2 \cdot \xi - v_1 \cdot \xi)^2 \\ 0 \\ u_2 \\ -\rho_2 b \\ \frac{v_2 \cdot \xi - v_1 \cdot \xi}{\xi^T P_2 \xi - (v_2 \cdot \xi - v_1 \cdot \xi)^2} [3\xi^T P_2 \xi - (v_2 \cdot \xi - v_1 \cdot \xi)^2 - 2P_2] b \xi \\ \frac{1}{(v_2 \cdot \xi - v_1 \cdot \xi)^2 - \xi^T P_2 \xi} D_2(\xi) [3\xi^T P_2 \xi - (v_2 \cdot \xi - v_1 \cdot \xi)^2 - 2P_2] b \xi \end{pmatrix}$$

where $u_1 \neq 0$, $C(\xi)u_1 = 0$, and

$$C(\xi)u_2 = \left[\rho_2 b^2 - [\rho_2 a_2 + 3\xi^T P_2 \xi - (v_2 \cdot \xi - v_1 \cdot \xi)^2] \left(a_1 + \frac{P_1}{\rho_1} \right) \right] \xi.$$

Note that $C(\xi)$ is a matrix of size $d(d+1)/2 \times d$ and of rank d , hence it is onto and the dimension of its kernel is $d(d-1)/2$.

We can also find $d-1$ eigenvectors corresponding to the eigenvalue $v_1 \cdot \xi \pm \sqrt{\xi^T P_1 \xi}$, given by

$$X_3 := \begin{pmatrix} 0 \\ \xi^\perp \\ \pm \frac{1}{\sqrt{\xi^T P_1 \xi}} D(\xi) \xi^\perp \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{where } \xi^\perp \neq 0 \quad \text{and} \quad \xi^T \xi^\perp = 0.$$

Since the system is invariant under permutation of indices $1 \leftrightarrow 2$, one can find similar eigenvectors for the eigenvalues $v_2 \cdot \xi$ and $v_2 \cdot \xi \pm \sqrt{\xi^T P_2 \xi}$. Hence $C(Y, \xi)$ is diagonalizable and System (2.54) is hyperbolic. $\square$

4.3. Other systems

In the previous sections, we showed that the presence of turbulence in a two-phase flow, modeled by a Reynolds tensor, enables to obtain a hyperbolic system coming from a variational principle. We showed that the turbulence can be added in various ways and gave three different examples. Theorem 2.4 states that the three examples given previously are the only ones for which the system of equations obtained is hyperbolic. Recall the statement of Theorem 2.4:

Theorem 2.5. *Consider the following general Lagrangian density:*

$$\mathcal{L} := \frac{1}{2} \rho_1 |v_1|^2 + \frac{1}{2} \rho_2 |v_2|^2 - \rho_1 \mathcal{E}_1 \left(\frac{\rho_1}{\alpha_1} \right) - \rho_2 \mathcal{E}_2 \left(\frac{\rho_2}{\alpha_2} \right) - \frac{1}{2} r \text{Tr}(P), \quad (2.56)$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 4. Other models

endowed with the constraints

$$\partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0, \quad i = 1, 2 \quad (2.57)$$

and

$$\partial_t P + (u \cdot \nabla) P + \frac{\partial u}{\partial x} P + P \left(\frac{\partial u}{\partial x} \right)^T = 0, \quad (2.58)$$

where

$$(r, u) \in \{\rho_1, \rho_2, \rho\} \times \{v_1, v_2, v\}.$$

Let us consider the system formed by Equations (2.57), (2.58) and the Euler-Lagrange equations associated to the Lagrangian density (2.56). We denote by M the matrix of this system. Then, in dimension 1, all the eigenvalues of M are real when $v_1 = v_2$, whatever the choice of convex internal energies $\rho \mathcal{E}_i(\rho)$, $i = 1, 2$, if and only if the density r and the velocity u are compatible, i.e. the following equation is satisfied:

$$\partial_t r + \operatorname{div}(r u) = 0.$$

Proof. We treat each possible value of $(r, u) \in \{\rho_1, \rho_2, \rho\} \times \{v_1, v_2, v\}$ separately. The case $(r, u) = (\rho, v)$ was treated in the study of System (2.10). The case $(r, u) = (\rho_2, v_2)$ was treated in the study of System (2.52). By symmetry, we are left to study the three following cases:

- $(r, u) = (\rho_1, v_2)$
- $(r, u) = (\rho, v_2)$
- $(r, u) = (\rho_1, v)$

4.3.1. Case $(r, u) = (\rho_1, v_2)$

We obtain the following system:

$$\begin{aligned} \partial_t \rho_i + \operatorname{div}(\rho_i v_i) &= 0, \quad i = 1, 2, \\ \partial_t v_1 + (v_1 \cdot \nabla) v_1 + \nabla(U_1 + \frac{1}{2} \operatorname{Tr} P) &= 0, \\ \partial_t v_2 + (v_2 \cdot \nabla) v_2 + \nabla(U_2) + \frac{1}{\rho_2} \operatorname{div}(\rho_1 P)^T - \frac{\rho_1}{2\rho_2} \nabla(\operatorname{Tr} P) &= 0, \\ \partial_t P + (v_2 \cdot \nabla) P + \frac{\partial v_2}{\partial x} P + P \left(\frac{\partial v_2}{\partial x} \right)^T &= 0. \end{aligned}$$

In 1d, the characteristic polynomial is given by

$$\begin{aligned} \chi(\lambda) = & (v_2 - \lambda) \left[(v_1 - \lambda)^2 (v_2 - \lambda)^2 - \rho_1 a_1 (v_2 - \lambda)^2 \right. \\ & \left. - \rho_2 \left(a_2 + \rho_1 \frac{P}{\rho_2^2} \right) (v_1 - \lambda)^2 + \rho_1 \rho_2 \left[a_1 \left(a_2 + \rho_1 \frac{P}{\rho_2^2} \right) - \left(b + \frac{P}{\rho_2} \right)^2 \right] \right]. \end{aligned}$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 4. Other models

Let us look at the case $v_1 = v_2 = v$. If we denote $X := (v - \lambda)^2$, we get that

$$\frac{\chi(\lambda)}{v - \lambda} = X^2 - \left(\rho_1 a_1 + \rho_2 a_2 + \frac{\rho_1 P}{\rho_2} \right) X + \rho_1 \rho_2 \left[a_1 \left(a_2 + \frac{\rho_1 P}{\rho_2^2} \right) - \left(b + \frac{P}{\rho_2} \right)^2 \right] =: Q(X).$$

The discriminant of $Q(X)$ is

$$\begin{aligned} \Delta &:= \left(\rho_1 a_1 + \rho_2 a_2 + \frac{\rho_1 P}{\rho_2} \right)^2 - 4 \rho_1 \rho_2 \left[a_1 \left(a_2 + \frac{\rho_1 P}{\rho_2^2} \right) - \left(b + \frac{P}{\rho_2} \right)^2 \right] \\ &= \left(\rho_1 a_1 - \rho_2 a_2 - \frac{\rho_1 P}{\rho_2} \right)^2 + 4 \rho_1 \rho_2 \left(b + \frac{P}{\rho_2} \right)^2 > 0. \end{aligned}$$

Hence Q has two real roots. This gives four roots for χ . In order that the roots of χ be real, the roots of Q have to be positive. This gives the condition

$$0 < a_1 \left(a_2 + \frac{\rho_1 P}{\rho_2^2} \right) - \left(b + \frac{P}{\rho_2} \right)^2 = \frac{P}{\rho_2^2} (\rho_1 a_1 - 2 \rho_2 b - P).$$

Hence there are some admissible values of a_1, b, P such that χ has complex roots.

4.3.2. Case $(r, u) = (\rho, v_2)$

We obtain the following system:

$$\begin{aligned} \partial_t \rho_i + \operatorname{div}(\rho_i v_i) &= 0, \quad i = 1, 2, \\ \partial_t v_1 + (v_1 \cdot \nabla) v_1 + \nabla(U_1 + \frac{1}{2} \operatorname{Tr} P) &= 0, \\ \partial_t v_2 + (v_2 \cdot \nabla) v_2 + \nabla(U_2) + \frac{1}{\rho_2} \operatorname{div}(\rho P)^T - \frac{\rho_1}{2 \rho_2} \nabla(\operatorname{Tr} P) &= 0, \\ \partial_t P + (v_2 \cdot \nabla) P + \frac{\partial v_2}{\partial x} P + P \left(\frac{\partial v_2}{\partial x} \right)^T &= 0. \end{aligned}$$

In 1d, the characteristic polynomial is given by

$$\begin{aligned} \chi(\lambda) &= (v_2 - \lambda) \left[(v_1 - \lambda)^2 (v_2 - \lambda)^2 - \rho_1 a_1 (v_2 - \lambda)^2 \right. \\ &\quad \left. - \rho_2 \left[a_2 + (\rho_1 + 3 \rho_2) \frac{P}{\rho_2^2} \right] (v_1 - \lambda)^2 + \rho_1 \rho_2 \left[a_1 \left(a_2 + (\rho_1 + 3 \rho_2) \frac{P}{\rho_2^2} \right) - \left(b + \frac{P}{\rho_2} \right)^2 \right] \right]. \end{aligned}$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 4. Other models

Let us look at the case $v_1 = v_2 = v$. If we denote $X := (v - \lambda)^2$, we get that

$$\begin{aligned} \frac{\chi(\lambda)}{v - \lambda} &= X^2 - \left(\rho_1 a_1 + \rho_2 a_2 + (\rho_1 + 3\rho_2) \frac{P}{\rho_2^2} \right) X \\ &\quad + \rho_1 \rho_2 \left[a_1 \left(a_2 + (\rho_1 + 3\rho_2) \frac{P}{\rho_2^2} \right) - \left(b + \frac{P}{\rho_2} \right)^2 \right] \\ &=: Q(X). \end{aligned}$$

The discriminant of $Q(X)$ is

$$\begin{aligned} \Delta &:= \left(\rho_1 a_1 + \rho_2 a_2 + (\rho_1 + 3\rho_2) \frac{P}{\rho_2^2} \right)^2 - 4\rho_1 \rho_2 \left[a_1 \left(a_2 + (\rho_1 + 3\rho_2) \frac{P}{\rho_2^2} \right) - \left(b + \frac{P}{\rho_2} \right)^2 \right] \\ &= \left(\rho_1 a_1 - \rho_2 a_2 - (\rho_1 + 3\rho_2) \frac{P}{\rho_2^2} \right)^2 + 4\rho_1 \rho_2 \left(b + \frac{P}{\rho_2} \right)^2 > 0. \end{aligned}$$

Hence Q has two real roots. This gives four roots for χ . In order that the roots of χ be real, the roots of Q have to be positive. This gives the condition

$$0 < a_1 \left(a_2 + (\rho_1 + 3\rho_2) \frac{P}{\rho_2^2} \right) - \left(b + \frac{P}{\rho_2} \right)^2 = \frac{P}{\rho_2^2} [(\rho_1 + 3\rho_2)a_1 - 2\rho_2 b - P].$$

Hence there are some admissible values of a_1, b, P such that χ has complex roots.

4.3.3. Case $(r, u) = (\rho_1, v)$

We obtain the following system:

$$\begin{aligned} \partial_t \rho_i + \operatorname{div}(\rho_i v_i) &= 0, \quad i = 1, 2, \\ \partial_t v_1 + (v_1 \cdot \nabla) v_1 + \operatorname{curl}(v_2) \wedge (v - v_1) + \nabla(U_1) + \frac{\rho_2}{2\rho} \nabla(\operatorname{Tr} P) + \frac{1}{\rho} \operatorname{div}(\rho_1 P)^T &= 0, \\ \partial_t v_2 + (v_2 \cdot \nabla) v_2 + \operatorname{curl}(v_2) \wedge (v - v_2) + \nabla(U_2) - \frac{\rho_1}{2\rho} \nabla(\operatorname{Tr} P) + \frac{1}{\rho} \operatorname{div}(\rho_1 P)^T &= 0, \\ \partial_t P + (v \cdot \nabla) P + \frac{\partial v}{\partial x} P + P \left(\frac{\partial v}{\partial x} \right)^T &= 0. \end{aligned}$$

In $1d$, the characteristic polynomial is given by

$$\begin{aligned} \chi(\lambda) &= \left[(v_1 - \lambda)^2 (v_2 - \lambda)^2 - \rho_1 \left(a_1 + 3 \frac{\rho_1}{\rho^2} P + 2 \frac{\rho_2}{\rho^2} P \right) (v_2 - \lambda)^2 - \rho_2 \left(a_2 + \rho_1 \frac{P}{\rho^2} \right) (v_1 - \lambda)^2 \right. \\ &\quad \left. + \rho_1 \rho_2 \left[\left(a_1 + 3 \frac{\rho_1}{\rho^2} P + 2 \frac{\rho_2}{\rho^2} P \right) \left(a_2 + \rho_1 \frac{P}{\rho^2} \right) - \left(b + (2\rho_1 + \rho_2) \frac{P}{\rho^2} \right)^2 \right] \right] (v - \lambda). \end{aligned}$$

Let us look at the case $\nu_1 = \nu_2 = \nu$. If we denote $X := (\nu - \lambda)^2$, we get that

$$\begin{aligned} \frac{\chi(\lambda)}{\nu - \lambda} = & X^2 - \left[\rho_1 \left(a_1 + 3 \frac{\rho_1}{\rho^2} P + 2 \frac{\rho_2}{\rho^2} P \right) + \rho_2 \left(a_2 + \rho_1 \frac{P}{\rho^2} \right) \right] X \\ & + \rho_1 \rho_2 \left[\left(a_1 + 3 \frac{\rho_1}{\rho^2} P + 2 \frac{\rho_2}{\rho^2} P \right) \left(a_2 + \rho_1 \frac{P}{\rho^2} \right) - \left(b + (2\rho_1 + \rho_2) \frac{P}{\rho^2} \right)^2 \right] =: Q(X). \end{aligned}$$

The discriminant of $Q(X)$ is

$$\begin{aligned} \Delta := & \left[\rho_1 \left(a_1 + 3 \frac{\rho_1}{\rho^2} P + 2 \frac{\rho_2}{\rho^2} P \right) + \rho_2 \left(a_2 + \rho_1 \frac{P}{\rho^2} \right) \right]^2 \\ & - 4 \rho_1 \rho_2 \left[\left(a_1 + 3 \frac{\rho_1}{\rho^2} P + 2 \frac{\rho_2}{\rho^2} P \right) \left(a_2 + \rho_1 \frac{P}{\rho^2} \right) - \left(b + (2\rho_1 + \rho_2) \frac{P}{\rho^2} \right)^2 \right] \\ = & \left[\rho_1 \left(a_1 + 3 \frac{\rho_1}{\rho^2} P + 2 \frac{\rho_2}{\rho^2} P \right) - \rho_2 \left(a_2 + \rho_1 \frac{P}{\rho^2} \right) \right]^2 + 4 \rho_1 \rho_2 \left(b + (2\rho_1 + \rho_2) \frac{P}{\rho^2} \right)^2 > 0. \end{aligned}$$

Hence Q has two real roots. This gives four roots for χ . In order that the roots of χ be real, the roots of Q have to be positive. This gives the condition

$$\begin{aligned} 0 < & \left(a_1 + 3 \frac{\rho_1}{\rho^2} P + 2 \frac{\rho_2}{\rho^2} P \right) \left(a_2 + \rho_1 \frac{P}{\rho^2} \right) - \left(b + (2\rho_1 + \rho_2) \frac{P}{\rho^2} \right)^2 \\ < & \frac{P}{\rho^2} (\rho_1 (a_1 + a_2 - 2b) + 2\rho (a_2 - b) - P). \end{aligned}$$

Hence there are some admissible values of a_1, a_2, b, P such that χ has complex roots. $\square$

5. Appendix

5.1. Computation of the characteristic polynomial

We compute here the characteristic polynomial of the matrix $A(Y, \xi)$ given at the beginning of Section 3, in order to get the eigenvalues of this matrix. We first state two lemmas which will be useful in the proof.

Lemme 3. *For any $\xi \in \mathbb{R}^d$, one has $C(\xi)D(\xi) = (\xi^T P \xi) I_d + P \xi \xi^T$, where $C(\xi)$ and $D(\xi)$ are the matrices defined in the beginning of Section 3.*

The proof of this lemma can be obtained by a direct computation. The second lemma is also elementary:

Lemme 4. *Let $x, y \in \mathbb{R}^d$. Consider the matrix $B := I_d + xy^T$. Then the determinant of B is given by $\det(B) = 1 + x \cdot y$. Furthermore, when $\det(B) \neq 0$,*

$$B^{-1} = I_d - \frac{1}{1 + x \cdot y} xy^T.$$

Proof. By density and continuity of the determinant, we can restrict ourselves to the case $x \cdot y \neq 0$. We see that the linear space $y^\perp$ is a subspace of the eigenspace of B associated to the eigenvalue 1, of dimension $d - 1$. Furthermore, x is an eigenvector of B for the eigenvalue $1 + x \cdot y$. Hence B is diagonalizable and $\det(B) = 1 + x \cdot y$. The formula for B^{-1} can be checked by computing the product BB^{-1} . $\square$

We will proceed by using the Gaussian elimination method to obtain an upper block triangular matrix. Since the matrix $A(Y, \xi)$ is quite large, we will perform the operations on the blocks rather than the coefficients. When doing this, one has to be careful that blocks generally do not commute with each other. Hence it is important to keep in mind that operations on rows of blocks are made through left multiplication, while operations on block columns are performed through right multiplication. By definition, the characteristic polynomial of $A(Y, \xi)$ is given by

$$\chi_{A(Y, \xi)}(\lambda) := \begin{vmatrix} v_2 \cdot \xi - \lambda & \rho_2 \xi^T & 0 & 0 & 0 \\ \alpha \xi & (v \cdot \xi - \lambda)I_d + \xi(v_2 - v)^T & (\beta + P/\rho)\xi & 0 & C(\xi) \\ 0 & 0 & v \cdot \xi - \lambda & \rho \xi^T & 0 \\ \delta \xi & 0 & (\gamma + P/\rho)\xi & (v \cdot \xi - \lambda)I_d & C(\xi) \\ 0 & 0 & 0 & D(\xi) & (v \cdot \xi - \lambda)I_d \end{vmatrix}.$$

We multiply the last block column by $-(v \cdot \xi - \lambda)^{-1}D(\xi)$ and add it to the fourth block column. By Lemma 3, the characteristic polynomial is equal to:

$$\chi_{A(Y, \xi)}(\lambda) = \begin{vmatrix} v_2 \cdot \xi - \lambda & \rho_2 \xi^T & 0 & 0 & 0 \\ \alpha \xi & (v \cdot \xi - \lambda)I_d + \xi(v_2 - v)^T & (\beta + P/\rho)\xi & N_0 & C(\xi) \\ 0 & 0 & v \cdot \xi - \lambda & \rho \xi^T & 0 \\ \delta \xi & 0 & (\gamma + P/\rho)\xi & M_0 & C(\xi) \\ 0 & 0 & 0 & 0 & (v \cdot \xi - \lambda)I_d \end{vmatrix}. \quad (2.59)$$

where we defined

$$M_0 := \frac{1}{v \cdot \xi - \lambda} \left([(v \cdot \xi - \lambda)^2 - \xi^T P \xi] I_d - P \xi \xi^T \right),$$

and

$$N_0 := \frac{-1}{v \cdot \xi - \lambda} \left[(\xi^T P \xi) I_d + P \xi \xi^T \right].$$

We now multiply the third block column of (2.59) by $-\rho \xi^T / (v \cdot \xi - \lambda)$ and add it to the fourth block column. We obtain that

$$\chi_{A(Y, \xi)}(\lambda) = \begin{vmatrix} v_2 \cdot \xi - \lambda & \rho_2 \xi^T & 0 & 0 & 0 \\ \alpha \xi & (v \cdot \xi - \lambda)I_d + \xi(v_2 - v)^T & (\beta + P/\rho)\xi & N_1 & C(\xi) \\ 0 & 0 & v \cdot \xi - \lambda & 0 & 0 \\ \delta \xi & 0 & (\gamma + P/\rho)\xi & M_1 & C(\xi) \\ 0 & 0 & 0 & 0 & (v \cdot \xi - \lambda)I_d \end{vmatrix}, \quad (2.60)$$

where

$$N_1 := \frac{-1}{v \cdot \xi - \lambda} [(\xi^T P \xi) I_d + 2P \xi \xi^T + \rho \beta \xi \xi^T], \quad (2.61)$$

and

$$\begin{aligned} M_1 &:= \frac{1}{v \cdot \xi - \lambda} [(v \cdot \xi - \lambda)^2 - \xi^T P \xi] I_d - 2P \xi \xi^T - \rho \gamma \xi \xi^T \\ &= \frac{(v \cdot \xi - \lambda)^2 - \xi^T P \xi}{v \cdot \xi - \lambda} \left[I_d - \frac{1}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi} (2P + \rho \gamma) \xi \xi^T \right]. \end{aligned}$$

By Lemma 4, it follows that

$$\begin{aligned} M_1^{-1} &= \frac{v \cdot \xi - \lambda}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi} \left[I_d + \frac{1}{1 - \frac{2\xi^T P \xi + \rho \gamma}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi}} \frac{1}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi} (2P + \rho \gamma) \xi \xi^T \right] \\ &= \frac{v \cdot \xi - \lambda}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi} \left[I_d + \frac{1}{(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho \gamma} (2P + \rho \gamma) \xi \xi^T \right] \end{aligned} \quad (2.62)$$

(recall that $\xi^T \xi = 1$). We now multiply the fourth block column of (2.60) by $-M_1^{-1} \delta \xi$ to the right and add it to the first block column. We obtain that the characteristic polynomial is equal to the determinant of an upper block triangular matrix:

$$\chi_{A(Y, \xi)}(\lambda) = \begin{vmatrix} B_1 & B_2 \\ 0 & B_3 \end{vmatrix} = |B_1| |B_3|, \quad (2.63)$$

where

$$\begin{aligned} B_1 &:= \begin{pmatrix} v_2 \cdot \xi - \lambda & \rho_2 \xi^T \\ \alpha \xi - N_1 M_1^{-1} \delta \xi & (v \cdot \xi - \lambda) I_d + \xi (v_2 - v)^T \end{pmatrix}, \\ B_2 &:= \begin{pmatrix} 0 & 0 & 0 \\ (\beta + P/\rho) \xi & N_1 & C(\xi) \end{pmatrix}, \end{aligned}$$

and

$$B_3 := \begin{pmatrix} v \cdot \xi - \lambda & 0 & 0 \\ (\gamma + P/\rho) \xi & M_1 & C(\xi) \\ 0 & 0 & (v \cdot \xi - \lambda) I_d \end{pmatrix}.$$

In view of (2.63), we need to compute the determinants of the matrices B_1 and B_3 . Since B_3 is block triangular, its determinant is equal to the product of the determinants of each diagonal block. By Lemma 4,

$$\begin{aligned} |B_3| &= (v \cdot \xi - \lambda) |M_1| |(v \cdot \xi - \lambda) I_d| \\ &= (v \cdot \xi - \lambda) \left[\frac{(v \cdot \xi - \lambda)^2 - \xi^T P \xi}{v \cdot \xi - \lambda} \right]^d \left(1 - \frac{2\xi^T P \xi + \rho \gamma}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi} \right) (v \cdot \xi - \lambda)^{d(d+1)/2} \\ &= (v \cdot \xi - \lambda)^{d(d-1)/2+1} [(v \cdot \xi - \lambda)^2 - \xi^T P \xi]^{d-1} [(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho \gamma]. \end{aligned}$$

We now compute the determinant of B_1 . Let us define

$$E := (\nu \cdot \xi - \lambda)I_d + \xi(\nu_2 - \nu)^T = (\nu \cdot \xi - \lambda) \left[I_d + \frac{1}{\nu \cdot \xi - \lambda} \xi(\nu_2 - \nu)^T \right]. \quad (2.64)$$

By Lemma 4,

$$\begin{aligned} E^{-1} &= \frac{1}{\nu \cdot \xi - \lambda} \left[I_d - \frac{1}{1 + \frac{\nu_2 \cdot \xi - \nu \cdot \xi}{\nu \cdot \xi - \lambda}} \frac{1}{\nu \cdot \xi - \lambda} \xi(\nu_2 - \nu)^T \right] \\ &= \frac{1}{\nu \cdot \xi - \lambda} \left[I_d - \frac{1}{\nu_2 \cdot \xi - \lambda} \xi(\nu_2 - \nu)^T \right]. \end{aligned} \quad (2.65)$$

We now multiply the second block row of B_1 by $-\rho_2 \xi^T E^{-1}$ and add it to the first block row. We obtain that

$$\begin{aligned} |B_1| &= \begin{vmatrix} \nu_2 \cdot \xi - \lambda - \rho_2 \xi^T E^{-1} (\alpha \xi - N_1 M_1^{-1} \delta \xi) & 0 \\ \alpha \xi - N_1 M_1^{-1} \delta \xi & E \end{vmatrix} \\ &= [\nu_2 \cdot \xi - \lambda - \rho_2 \xi^T E^{-1} (\alpha \xi - N_1 M_1^{-1} \delta \xi)] |E|. \end{aligned}$$

By Lemma 4,

$$|E| = (\nu \cdot \xi - \lambda)^d \left(1 + \frac{\nu_2 \cdot \xi - \nu \cdot \xi}{\nu \cdot \xi - \lambda} \right) = (\nu_2 \cdot \xi - \lambda)(\nu \cdot \xi - \lambda)^{d-1}.$$

We are left to compute the scalar $[\nu_2 \cdot \xi - \lambda - \rho_2 \xi^T E^{-1} (\alpha \xi - N_1 M_1^{-1} \delta \xi)]$. First, by Equation (2.65),

$$\xi^T E^{-1} = \frac{1}{\nu \cdot \xi - \lambda} \xi^T - \frac{1}{(\nu \cdot \xi - \lambda)(\nu_2 \cdot \xi - \lambda)} (\nu_2 - \nu)^T.$$

It follows that

$$\xi^T E^{-1} \xi = \frac{1}{\nu_2 \cdot \xi - \lambda}.$$

On the other hand, Equation (2.62) gives that

$$M_1^{-1} \xi = \frac{\nu \cdot \xi - \lambda}{(\nu \cdot \xi - \lambda)^2 - \xi^T P \xi} \left[1 + \frac{1}{(\nu \cdot \xi - \lambda)^2 - 3 \xi^T P \xi - \rho \gamma} (2P + \rho \gamma) \right] \xi.$$

Equation (2.61) then yields

$$\begin{aligned}
 & N_1 M_1^{-1} \xi \\
 &= \frac{-1}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi} \left[\xi^T P \xi + \frac{\xi^T P \xi}{(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho\gamma} (2P + \rho\gamma) \right. \\
 &\quad \left. + (2P + \rho\beta) \left(1 + \frac{2\xi^T P \xi + \rho\gamma}{(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho\gamma} \right) \right] \xi \\
 &= \frac{-1}{(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho\gamma} \left[\xi^T P \xi \frac{(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi} + \frac{2(v \cdot \xi - \lambda)^2}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi} P + \rho\beta \right] \xi \\
 &= \frac{-1}{(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho\gamma} \left[\xi^T P \xi + \rho\beta + 2 \frac{(v \cdot \xi - \lambda)^2 P - (\xi^T P \xi)^2}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi} \right] \xi. \tag{2.66}
 \end{aligned}$$

Hence

$$\begin{aligned}
 & \xi^T E^{-1} N_1 M_1^{-1} \xi \\
 &= \frac{-1}{(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho\gamma} \left[\frac{\xi^T P \xi + \rho\beta}{v_2 \cdot \xi - \lambda} + \frac{2\xi^T P \xi}{v \cdot \xi - \lambda} \right. \\
 &\quad \left. - 2 \frac{(v \cdot \xi - \lambda)^2 (v_2 - v)^T P \xi - (\xi^T P \xi)^2 (v_2 - v)^T \xi}{(v \cdot \xi - \lambda)(v_2 \cdot \xi - \lambda)[(v \cdot \xi - \lambda)^2 - \xi^T P \xi]} \right] \\
 &= \frac{-1}{[(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho\gamma] (v_2 \cdot \xi - \lambda)} \\
 &\quad \times \left[3\xi^T P \xi + \rho\beta + \frac{2(v \cdot \xi - \lambda)(v_2 - v)^T [(\xi^T P \xi) - P] \xi}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi} \right],
 \end{aligned}$$

and

$$\begin{aligned}
 & v_2 \cdot \xi - \lambda - \rho_2 \xi^T E^{-1} (\alpha \xi - N_1 M_1^{-1} \delta \xi) \\
 &= v_2 \cdot \xi - \lambda - \frac{\alpha \rho_2}{v_2 \cdot \xi - \lambda} - \frac{\rho_2 \delta}{[(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho\gamma] (v_2 \cdot \xi - \lambda)} \\
 &\quad \times \left[3\xi^T P \xi + \rho\beta + \frac{2(v \cdot \xi - \lambda)(v_2 - v)^T [(\xi^T P \xi) - P] \xi}{(v \cdot \xi - \lambda)^2 - \xi^T P \xi} \right] \\
 &= \frac{[(v_2 \cdot \xi - \lambda)^2 - \alpha \rho_2] [(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho\gamma] - \rho_2 \delta (3\xi^T P \xi + \rho\beta)}{(v_2 \cdot \xi - \lambda) [(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho\gamma]} \\
 &\quad + \frac{2\rho_2 \delta (v \cdot \xi - \lambda)(v_2 - v)^T [P - (\xi^T P \xi)] \xi}{(v_2 \cdot \xi - \lambda) [(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho\gamma] [(v \cdot \xi - \lambda)^2 - \xi^T P \xi]}.
 \end{aligned}$$

Finally,

$$\begin{aligned}
 \chi_{A(Y, \xi)}(\lambda) &= (v \cdot \xi - \lambda)^{(d+2)(d-1)/2+1} [(v \cdot \xi - \lambda)^2 - \xi^T P \xi]^{d-2} \\
 &\quad \times [2\rho_2 \delta (v \cdot \xi - \lambda)(v_2 - v)^T [P - (\xi^T P \xi)] \xi + [(v \cdot \xi - \lambda)^2 - \xi^T P \xi] q(\lambda)],
 \end{aligned}$$

where

$$q(\lambda) =: [(v_2 \cdot \xi - \lambda)^2 - \alpha \rho_2] [(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho \gamma] - \rho_2 \delta (3\xi^T P \xi + \rho \beta),$$

as claimed in Proposition 2.1.

Case of the system without lift terms

When there are no lift terms the previous computations can be adapted to compute the characteristic polynomial of the modified system. Let us also denote $A'(Y, \xi)$ the matrix of System (2.16). It is given by

$$A'(Y, \xi) = \begin{pmatrix} v_2 \cdot \xi & \rho_2 \xi^T & 0 & 0 & 0 \\ \alpha \xi & (v_2 \cdot \xi) I_d & (\beta + P/\rho) \xi & 0 & C(\xi) \\ 0 & 0 & v \cdot \xi & \rho \xi^T & 0 \\ \delta \xi & 0 & (\gamma + P/\rho) \xi & (v \cdot \xi) I_d & C(\xi) \\ 0 & 0 & 0 & D(\xi) & (v \cdot \xi) I_d \end{pmatrix}.$$

Thus the only difference is that one should replace the matrix E of (2.64) by

$$E' := (v_2 \cdot \xi - \lambda) I_d, \quad \text{hence} \quad E^{-1} = \frac{1}{v_2 \cdot \xi - \lambda} I_d.$$

All computations of the proof stay unchanged until Equation (2.64). We deduce as in (2.63) that

$$\chi_{A'(Y, \xi)} = |B_1| |B_3|,$$

with

$$|B_1| = [v_2 \cdot \xi - \lambda - \rho_2 \xi^T E'^{-1} (\alpha \xi - N_1 M_1^{-1} \delta \xi)] |E'|$$

and

$$|B_3| = (v \cdot \xi - \lambda)^{d(d-1)/2+1} [(v \cdot \xi - \lambda)^2 - \xi^T P \xi]^{d-1} [(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho \gamma]$$

as previously, substituting E' to E . We compute that

$$|E'| = (v_2 \cdot \xi - \lambda)^d, \quad \xi^T E'^{-1} \xi = \frac{1}{v_2 \cdot \xi - \lambda}$$

and, with the help of (2.66),

$$\xi^T E'^{-1} N_1 M_1 \xi = - \frac{3\xi^T P \xi + \rho \beta}{[(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho \gamma] (v_2 \cdot \xi - \lambda)}.$$

Hence

$$|B_1| = (v_2 \cdot \xi - \lambda)^{d-1} \frac{[(v_2 \cdot \xi - \lambda)^2 - \rho_2 \alpha] [(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho \gamma] - \rho_2 \delta [3\xi^T P \xi + \rho \beta]}{(v \cdot \xi - \lambda)^2 - 3\xi^T P \xi - \rho \gamma}.$$

2. Modèles hyperboliques d'écoulements diphasiques turbulents – 5. Appendix

Finally, the characteristic polynomial is equal to

$$\chi_{A'(Y,\xi)} = (\nu_2 \cdot \xi - \lambda)^{d-1} (\nu \cdot \xi - \lambda)^{d(d-1)/2+1} \left[(\nu \cdot \xi - \lambda)^2 - \xi^T P \xi \right]^{d-1} q(\lambda),$$

where

$$q(\lambda) := \left[(\nu_2 \cdot \xi - \lambda)^2 - \rho_2 \alpha \right] \left[(\nu \cdot \xi - \lambda)^2 - 3 \xi^T P \xi - \rho \gamma \right] - \rho_2 \delta [3 \xi^T P \xi + \rho \beta].$$

Deuxième partie

Modèles d'avalanche

3. Un modèle moyenné sur la profondeur pour les écoulements granulaires consistant avec la rhéologie $\mu(I)$ incompressible

Dans ce chapitre, on dérive un modèle moyenné sur la profondeur consistant avec la rhéologie $\mu(I)$ pour un écoulement granulaire incompressible sur plan incliné. Les deux premières variables du modèle sont la profondeur et la vitesse moyenne. Le cisaillement est aussi pris en compte via une troisième variable appelée enstrophie. Le système obtenu est un système hyperbolique de lois de conservation, admettant une équation supplémentaire pour l'énergie. Le système est dérivé à partir d'un développement asymptotique des variables de l'écoulement en puissances du paramètre de couche mince. Cette méthode garantit la consistance du modèle avec la rhéologie. Le profil de vitesse est un profil de Bagnold à l'ordre dominant et la correction du premier ordre de ce profil peut être calculée pour des écoulements qui ne sont pas stationnaires uniformes. La correction du premier ordre à la loi de friction granulaire classique est également écrite de manière consistante. Par conséquent, le seuil d'instabilité pour un écoulement stationnaire et uniforme est le même pour le modèle moyenné sur la profondeur et pour les équations de la rhéologie $\mu(I)$. De plus, une version du modèle d'ordre supérieur qui contient des termes diffusifs est également présentée. Le taux de croissance spatial, la vitesse de phase et la fréquence de coupure de la version avec diffusion sont en bon accord avec les données expérimentales et les prédictions théoriques de la rhéologie. La structure mathématique des équations permet d'utiliser des schémas numériques réputés et stables. Des simulations d'ondes de surface granulaires sont présentées. Le modèle a les mêmes limites que la rhéologie $\mu(I)$, en particulier pour les transitions solide/liquide et liquide/gaz, et nécessite donc une régularisation pour ces transitions. Une telle régularisation sera proposée au chapitre 4. Le chapitre 3 est tiré de l'article (DELEAGE et RICHARD 2025) :

Deleage, E., & Richard, G. L. (2025). A depth-averaged model for granular flow consistent with the incompressible rheology. *Journal of Fluid Mechanics*, 1009, A57.

1. Introduction

The concept of granular fluid can be used in order to model the flow of a large quantity of solid particles (see Andreotti et al. 2013). In dense regimes, the particles tend to stay in contact with each other and a macroscopic behaviour can be observed. This kind of flow can be found in many geophysical contexts, for which, in the presence of topography, the flow is driven by gravity. Two examples are landslides or dry snow avalanches. Modelling the flow of granular media with a good accuracy is thus of crucial importance in a large range of applications.

Granular flows exhibit complex behaviours that are distinct from more “classical” flows. In the context of flow down an inclined plane, a way to model these particular aspects is to use an adequate friction law. Granular friction laws were first introduced at the depth-averaged level. For numerical applications, depth-averaged models present the advantage of a reduced computation time. A first attempt to model granular flows specifically was proposed in 1989 by Savage et al. 1989 using a solid friction law. This model has some similarities with older models of dense snow avalanches, which also added an empirical fluid friction term. The approach of Savage et al. 1989 was extended to two dimensions (2D) and to complex topographies and applied to many cases of geophysical flows (e.g. Denlinger et al. 2001; Pitman et al. 2003; Pudasaini et al. 2003; Bouchut and Westdickenberg 2004; Denlinger et al. 2004; Mangeney-Castelnau, Bouchut, et al. 2005; Larrieu et al. 2006; Peruzzetto et al. 2019).

As it became clear that granular flows could not always be described by a solid friction law, a more elaborate friction law was proposed. An empirical approach was used by Pouliquen 1999b to derive a friction law based on experimental observations, still at the depth-averaged level. This law was refined by Pouliquen and Forterre 2002 who also proposed a non-monotonic law for the friction coefficient, distinguishing between dynamic and static regimes, as well as an intermediate regime taken into account by an extrapolation function. Contrary to the assumption of constant solid friction, these friction laws are compatible with the existence of steady uniform flows over a range of inclination angles for granular flows down a rough incline. The non-monotonic friction law is able to capture frictional hysteresis.

A specific granular rheology was then presented (GdR MiDi 2004), in which the friction coefficient μ is a function of a dimensionless parameter I , called inertial number. The inertial number is interpreted as a ratio between two time-scales related to the microscopic structure of the granular media (GdR MiDi 2004). This $\mu(I)$ rheology was then generalized into a constitutive law at the bulk level, for two-dimensional (2D) flows (GdR MiDi 2004; Pouliquen, Cassar, et al. 2005; Jop et al. 2005) and three-dimensional (3D) flows (Jop et al. 2006; Forterre 2006). The basal friction law of the model of Pouliquen and Forterre 2002 can be reinterpreted in the context of the $\mu(I)$ rheology (Andreotti et al. 2013). Extensions of the depth-averaged model to the 2D case on complex topography with applications to geophysical flows were proposed with the friction law of Pouliquen 1999b (Mangeney-Castelnau, Vilotte, et al. 2003) or with the non-monotonic friction law of Pouliquen and Forterre 2002 (Mangeney et al. 2007).

3. Modèle consistant d'écoulement granulaire incompressible – 1. Introduction

Roll waves for dense granular flows were experimentally observed by Forterre and Pouliquen [2003](#), who also measured the phase velocity and the spatial growth rate. This instability occurs above a critical Froude number and disappears above a cut-off frequency. This cut-off frequency does not appear in models that only take into account friction effects. By adding a diffusive term to the depth-averaged model, Forterre [2006](#) and Gray and Edwards [2014](#) derived models capable of predicting the existence of this cut-off frequency in agreement with the experimental data of Forterre and Pouliquen [2003](#). The spatial growth rate is also accurately predicted.

The diffusive model of Gray and Edwards [2014](#) was used with the friction coefficient of Pouliquen and Forterre [2002](#) in the dynamic regime to model granular roll waves (Razis et al. [2014](#)) and dense granular avalanches (Baker, Barker, et al. [2016](#)), as well as with the non-monotonic friction law (Pouliquen and Forterre [2002](#); Edwards, Viroulet, et al. [2017](#)) to model erosion-deposition waves and the formation of levees (Edwards and Gray [2015](#); Edwards, Viroulet, et al. [2017](#); Rocha et al. [2019](#)). The importance of the diffusive term in the depth-averaged model was highlighted by Baker, Barker, et al. [2016](#) and Rocha et al. [2019](#) and also by Baker, C. Johnson, et al. [2016](#) for segregation-induced finger formation in granular flows.

Despite the many successes of the $\mu(I)$ rheology in describing granular flows, it has been shown by Barker, Schaeffer, Bohórquez, et al. [2015](#) that this rheology is well-posed for intermediate inertial numbers, but can be ill-posed for low or high inertial numbers. This ill-posed behaviour is due to a short-wavelength instability with an infinite growth rate in the infinite wavenumber limit. Modifications to the $\mu(I)$ rheology involving compressibility were proposed by Heyman et al. [2017](#), Barker, Schaeffer, Shearer, et al. [2017](#) and Schaeffer et al. [2019](#) to solve this problem (see also the remarks of Goddard et al. [2018](#)). Further work on this subject was provided by Goddard et al. [2017](#).

In the case of thin layers, depth-averaged models can be derived by introducing a small parameter ε , which is a ratio between a vertical and a horizontal length scale, defining thus a shallow flow, and by using an asymptotic method with a systematic expansion of the variables in the governing equations as power series of the parameter ε . The leading order, or zeroth order in ε , governs the steady uniform flows. To model flows that are not stationary and uniform, it is important to ensure model consistency up to the first order in ε . The definition of the consistency of a model (see Richard, Gisclon, et al. [2019](#)) is relative to a set of hypotheses concerning the order of magnitude of the dimensionless numbers of the problem. In the case of granular flows down an incline, these numbers are the Froude number and the slope. Both are of $O(1)$ with respect to the small parameter ε , since neither is particularly small in the case of granular flows studied in the laboratory or encountered in nature. A model can always be written in such a way that the dominant terms are of $O(1)$. A model written in this way is said to be consistent of order n if, after substituting the model variables with their asymptotic expansions, all the terms disappear except for an $O(\varepsilon^{n+1})$ remainder.

Model consistency means that the approximation used to derive the model is consistent. In practice only the first-order consistency is really important. Therefore the consistency implies that no term of first order is neglected. From a more physical point

3. Modèle consistant d'écoulement granulaire incompressible – 1. Introduction

of view, the relations obtained for steady uniform flows are not exactly valid for variable flows. Consistency allows us to calculate the first-order corrections that appear for variable flows. These corrections result in terms of order 1 in the model. One of the consequences of writing an inconsistent model is that the stability threshold of the Kapitza instability predicted by the model differs from the threshold corresponding to the rheology in the long-wave limit. Consistency, on the other hand, ensures that the long-wave instability threshold predicted by the model and by rheology coincides exactly.

This asymptotic method was already used to derive consistent depth-averaged models for Newtonian fluids (e.g. Ruyer-Quil et al. 2000; Richard, Ruyer-Quil, et al. 2016, power-law fluids (Noble et al. 2013) and viscoplastic flows (Denisenko, Richard, et al. 2023), but no depth-averaged model of dense granular flow consistent with the incompressible $\mu(I)$ rheology has been derived. As a consequence, the instability threshold of steady uniform flows of the existing models (e.g. Forterre and Pouliquen 2003; Gray and Edwards 2014) is different from the theoretical one that was calculated by Forterre 2006 with the $\mu(I)$ rheology in the long-wave limit, because the bottom friction has been estimated from steady uniform flow relations, so that the first-order correction in ε has been neglected.

Another inconsistency is due to the treatment of the depth-averaged value $\langle u^2 \rangle$ of the square of the velocity u in the momentum flux. Since this flux term is already of first-order in ε , it can be estimated, for a first-order model, from the zero-order expression of u (steady uniform flows relations). If a no-slip condition is assumed at the bottom, the flow is strongly sheared and shear effects cannot be consistently neglected in the depth-averaged model. A first approach that has been used by Forterre and Pouliquen 2003 and Gray and Edwards 2014 is to introduce a parameter α , called shape factor, which depends on the velocity profile, writing $\langle u^2 \rangle = \alpha \langle u \rangle^2$. For a sheared flow, the shape factor is always greater than 1. In the case of steady and uniform granular flows, the velocity profile is a Bagnold profile (Andreotti et al. 2013) and the shape factor is equal to 5/4. If $\alpha = 5/4$, the model strongly overestimates the instability threshold of the Kapitza instability (Forterre 2006). The agreement is better if a constant velocity profile is assumed ($\alpha = 1$). The inconsistency of existing models therefore leads either to very significant deviations from predictions of the $\mu(I)$ rheology if the Bagnold profile is taken into account, or, paradoxically, to more reasonable (but nonetheless significant) deviations if the velocity profile of equilibrium granular flows is not taken into account. This shortcoming is directly attributable to the inconsistency of the derivation. A consistent model exactly predicts the instability threshold corresponding to the rheology while taking into account a Bagnold velocity profile for a steady uniform flow. Consistency at the first order of accuracy is sufficient to obtain these properties. Second-order terms are only useful to include diffusive effects in the model since diffusion is absent in a first-order model.

An additional difficulty is that the mathematical structure of the model is not satisfactory if $\alpha \neq 1$, because there is no energy balance equation for the system. The compatibility with an energy balance equation implies $\alpha = 1$. The proof of Richard and Gavrilyuk 2012 can be used straightforwardly for the model of Gray and Edwards

2014. It is usually better to take $\alpha = 1$, thus loosing the consistency for a sheared flow.

A strategy for taking into account a depth-varying velocity profile in a depth-averaged model in a consistent and well-posed way is to take the shear into account with an additional depth-averaged variable $\langle (u - \langle u \rangle)^2 \rangle / h^2$ (where h is the depth and $\langle \cdot \rangle$ the depth-averaging operator), called enstrophy because it is related to the square of the vorticity. This approach was initiated by Teshukov 2007 and developed by Richard and Gavriluk 2012 and Richard, Ruyer-Quil, et al. 2016 in particular. The depth-averaged model is thus a system of three equations which admits, by construction, an energy balance equation. The variables are the depth of the flow, the depth-averaged velocity, which are the usual variables for shallow-water equations, and the enstrophy. In one dimension (1D) and in the absence of diffusion, the obtained model is a hyperbolic system of conservation laws analogous to the compressible Euler equations. This enables to use well-known reliable and stable numerical schemes. This approach is quite general and has been used to derive depth-averaged models in various contexts, see for instance Richard, Ruyer-Quil, et al. 2016, Denisenko, Richard, et al. 2023, Richard 2024.

The motivation of this work is to derive a depth-averaged, first-order consistent, well-posed model for granular flows down an inclined plane described by the standard $\mu(I)$ rheology, while taking into account a Bagnold velocity profile for steady uniform flows. The model must admit exactly an energy balance equation, both for physical and numerical reasons, since the stability of a numerical scheme implies preserving the energy balance in a discrete framework. Another advantage of the consistency of the derivation is the possibility of reconstructing the depth profile of velocity, from the values of the depth-averaged quantities calculated by the model. This type of reconstruction has been obtained in the case of turbulent open-channel flows (Richard, Couderc, et al. 2023) and for viscoplastic yield-stress fluids (Denisenko, Richard, et al. 2023).

The paper is organized as follows: in §2, we present the equations that govern the motion of the granular medium, as well as their dimensionless form in the long-wave limit. In §3, we derive explicitly the asymptotic expansions at orders 0 and 1 in ϵ . In §4, the depth-averaged model is derived. The model is analysed in §5 and is compared with experimental data of Forterre and Pouliquen 2003. An extended version of the model, that takes into account diffusive effects, is given in §5.4. A numerical resolution in the case of granular roll waves is presented in §5.6. The reconstruction of the velocity profile is obtained in §5.1 and an example is given in §5.6. The limitations of the model are explained in §6.

2. Governing equations

2.1. Constitutive equations

We study the flow of a granular fluid, which is described by its density ρ and its velocity $\mathbf{v}$. We assume that the fluid is made of solid particles. Hence the density is

3. Modèle consistant d'écoulement granulaire incompressible – 2. Governing equations

proportional to ϕ the volume fraction of the beads and is written

$$\rho = \rho_p \phi, \quad (3.1)$$

where the constant ρ_p is the density of the beads. We also suppose that the volume fraction ϕ is constant, which implies that the flow is incompressible. We assume that the fluid is flowing down an inclined plane that makes an angle θ with the horizontal. The motion of the fluid is described by the mass and momentum balance equations, which read in the incompressible case with constant density

$$\begin{aligned} \frac{\partial \rho \mathbf{v}}{\partial t} + \mathbf{div}(\rho \mathbf{v} \otimes \mathbf{v}) &= \mathbf{div} \boldsymbol{\sigma} + \rho \mathbf{g}, \\ \mathbf{div} \mathbf{v} &= 0, \\ \phi &= \phi_c \text{ constant.} \end{aligned} \quad (3.2)$$

The second order tensor $\boldsymbol{\sigma}$ is called the Cauchy stress tensor and can be written

$$\boldsymbol{\sigma} = -p \mathbf{I}_d + \boldsymbol{\tau}, \quad (3.3)$$

where $\mathbf{I}_d$ is the identity tensor. The quantity p is the pressure and $\boldsymbol{\tau}$ is the deviatoric stress tensor. It is given by the constitutive law

$$\boldsymbol{\tau} = \eta_e \dot{\boldsymbol{\gamma}}, \quad (3.4)$$

where

$$\dot{\boldsymbol{\gamma}} = \mathbf{grad} \mathbf{v} + (\mathbf{grad} \mathbf{v})^T \quad (3.5)$$

is the strain-rate tensor and η_e is an effective viscosity. In the isotropic case, η_e is a scalar. In order to define the effective viscosity, we introduce the inertial number I , which is a dimensionless parameter given by the relation (GdR MiDi 2004)

$$I = \frac{|\dot{\boldsymbol{\gamma}}| d}{\sqrt{p/\rho_p}}. \quad (3.6)$$

The quantity d is the diameter of a bead, and $|\dot{\boldsymbol{\gamma}}|$ is the norm of the second order tensor $\dot{\boldsymbol{\gamma}}$, defined by

$$|\dot{\boldsymbol{\gamma}}| = \sqrt{\frac{\dot{\boldsymbol{\gamma}} : \dot{\boldsymbol{\gamma}}}{2}}. \quad (3.7)$$

We also introduce the friction coefficient $\mu(I)$, which depends only on I . The dependence is experimentally determined with three constants μ_1 , μ_2 and I_m as (Jop et al. 2006)

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + I_m/I}. \quad (3.8)$$

The two constants μ_1 and μ_2 are limit cases: the value μ_2 is the maximal value of $\mu(I)$, obtained when $I \rightarrow \infty$, the value μ_1 is the minimal value, obtained when $I = 0$, and the

3. Modèle consistant d'écoulement granulaire incompressible – 2. Governing equations

granular fluid only flows when $|\tau| > \mu_1 p$. The existence of a maximum value for the friction coefficient is debated (see Holyoake et al. 2012; Barker and Gray 2017, but only the standard $\mu(I)$ rheology will be considered in this article. Note that this relation can be inverted to write

$$I = I_m \frac{\mu - \mu_1}{\mu_2 - \mu} \quad (3.9)$$

The expression of the effective viscosity is then

$$\eta_e = \mu(I) \frac{p}{|\dot{\gamma}|}. \quad (3.10)$$

We denote by z the coordinate in the direction normal to the inclined plane and by x the coordinate in the direction parallel to the inclined plane (see Figure 3.1). The components of the velocity $\mathbf{v}$ are u and w in the x -direction and in the z -direction respectively.

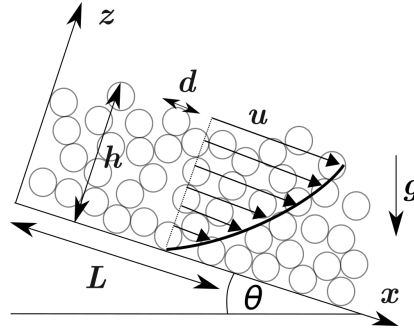


Figure 3.1. – Definition sketch.

The system (3.2) admits an energy balance equation. The mechanical energy density E of the fluid is equal to the sum of the kinetic energy density and the gravitational potential energy density:

$$E = \frac{1}{2} \rho |\mathbf{v}|^2 - \rho g x \sin \theta + \rho g z \cos \theta. \quad (3.11)$$

The energy balance equation is then

$$\frac{\partial E}{\partial t} + \text{div}(E \mathbf{v}) = \text{div}(\boldsymbol{\sigma} \cdot \mathbf{v}) - \frac{\boldsymbol{\sigma} : \dot{\boldsymbol{\gamma}}}{2}. \quad (3.12)$$

Using the incompressibility condition, Equation (3.12) can also be written

$$\frac{\partial E}{\partial t} + \text{div}[(E + p) \mathbf{v}] = \text{div}(\boldsymbol{\tau} \cdot \mathbf{v}) - \frac{\boldsymbol{\tau} : \dot{\boldsymbol{\gamma}}}{2}. \quad (3.13)$$

The granular fluid is assumed to be delimited downwards by the inclined plane at $z = 0$, and a free surface upwards at $z = h$. The field h is a function of (t, x) . For any field f depending on the variable z , we write $f(0) := f|_{z=0}$, and $f(h) := f|_{z=h}$. We

3. Modèle consistant d'écoulement granulaire incompressible – 2. Governing equations

denote by $\mathbf{n}(t, x)$ the unit vector normal to the free surface and pointing outward the flow. It is given by

$$\mathbf{n} = \frac{1}{\sqrt{1 + |\partial h / \partial x|^2}} \left(-\frac{\partial h}{\partial x}, 1 \right)^T. \quad (3.14)$$

We can now define the boundary conditions satisfied by the velocity:

- At $z = 0$, we assume a non-penetration condition given by

$$w(0) = 0, \quad (3.15)$$

as well as a no-slip condition, given by

$$u(0) = 0. \quad (3.16)$$

- At $z = h$, the velocity drives the evolution of the free surface via the kinematic boundary condition

$$\frac{\partial h}{\partial t} + u(h) \frac{\partial h}{\partial x} = w(h). \quad (3.17)$$

The dynamic boundary condition states that the normal stresses are continuous across the free surface. The normal stress above the free surface is given by the atmospheric pressure p_{atm} . Without loss of generality, this constant can be taken equal to zero. This leads to the equation

$$\boldsymbol{\sigma}(h) \cdot \mathbf{n} = 0, \quad (3.18)$$

which expands as

$$\left\{ \begin{array}{l} \tau_{xz}(h) + p(h) \frac{\partial h}{\partial x} - \tau_{xx}(h) \frac{\partial h}{\partial x} = 0, \\ \tau_{zz}(h) - p(h) - \tau_{xz}(h) \frac{\partial h}{\partial x} = 0. \end{array} \right. \quad (3.19a)$$

$$\left\{ \begin{array}{l} \tau_{xz}(h) + p(h) \frac{\partial h}{\partial x} - \tau_{xx}(h) \frac{\partial h}{\partial x} = 0, \\ \tau_{zz}(h) - p(h) - \tau_{xz}(h) \frac{\partial h}{\partial x} = 0. \end{array} \right. \quad (3.19b)$$

2.2. Dimensionless form of the equations

We write here all equations in a non-dimensional form. In order to do so, we denote, for any field f , a corresponding dimensionless field by $\tilde{f}$.

We assume here that the flow occurs at a typical horizontal scale L_0 , and a typical vertical scale h_0 , such that $\varepsilon = h_0 / L_0 \ll 1$ is a small parameter. We also assume that the horizontal velocity is of order u_0 , such that the Froude number

$$F = \frac{u_0}{\sqrt{g h_0}} \quad (3.20)$$

is of $O(1)$, as it is the case for instance in the experiments performed by Pouliquen [1999b](#) and Forterre and Pouliquen [2003](#). We introduce the following dimensionless

3. Modèle consistant d'écoulement granulaire incompressible – 2. Governing equations

variables:

$$\tilde{x} = \frac{x}{L_0}, \quad \tilde{z} = \frac{z}{h_0}, \quad \tilde{h} = \frac{h}{h_0}. \quad (3.21)$$

The time and velocity are scaled according to the usual shallow-water scaling as

$$\tilde{t} = t \frac{u_0}{L_0}, \quad \tilde{u} = \frac{u}{u_0}, \quad \tilde{w} = \frac{w}{\varepsilon u_0}, \quad (3.22)$$

The dimensionless pressure, density and size of the beads are found by

$$\tilde{p} = \frac{p}{\rho_p g h_0}, \quad \tilde{\rho} = \frac{\rho}{\rho_p} = \phi, \quad \tilde{d} = \frac{d}{h_0}. \quad (3.23)$$

We also scale the deviatoric stress tensor as

$$\tilde{\tau}_{xx} = \frac{\tau_{xx}}{\varepsilon \rho_p g h_0}, \quad \tilde{\tau}_{zz} = \frac{\tau_{zz}}{\varepsilon \rho_p g h_0}, \quad \tilde{\tau}_{xz} = \frac{\tau_{xz}}{\rho_p g h_0} \quad (3.24)$$

and the strain-rate tensor and energy as

$$|\tilde{\boldsymbol{\gamma}}| = |\dot{\boldsymbol{\gamma}}| \frac{h_0}{u_0}, \quad \tilde{E} = \frac{E}{\rho_p u_0^2}. \quad (3.25)$$

We obtain

$$\tilde{\tau}_{xx} = 2\mu(I) \frac{\tilde{p}}{|\tilde{\boldsymbol{\gamma}}|} \frac{\partial \tilde{u}}{\partial \tilde{x}}, \quad \tilde{\tau}_{zz} = 2\mu(I) \frac{\tilde{p}}{|\tilde{\boldsymbol{\gamma}}|} \frac{\partial \tilde{w}}{\partial \tilde{z}}, \quad \tilde{\tau}_{xz} = \mu(I) \frac{\tilde{p}}{|\tilde{\boldsymbol{\gamma}}|} \left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \varepsilon^2 \frac{\partial \tilde{w}}{\partial \tilde{x}} \right). \quad (3.26)$$

with

$$I = \frac{|\tilde{\boldsymbol{\gamma}}| \tilde{d}}{\sqrt{\tilde{p}}} F, \quad (3.27)$$

and

$$|\tilde{\boldsymbol{\gamma}}| = \left[\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \varepsilon^2 \frac{\partial \tilde{w}}{\partial \tilde{x}} \right)^2 + 2\varepsilon^2 \left(\frac{\partial \tilde{u}}{\partial \tilde{x}} \right)^2 + 2\varepsilon^2 \left(\frac{\partial \tilde{w}}{\partial \tilde{z}} \right)^2 \right]^{1/2}. \quad (3.28)$$

The dimensionless energy reads

$$\tilde{E} = \frac{1}{2} \phi (\tilde{u}^2 + \varepsilon^2 \tilde{w}^2) - \frac{\phi \sin \theta}{\varepsilon F^2} \tilde{x} + \frac{\phi \cos \theta}{F^2} \tilde{z}. \quad (3.29)$$

The divergence-free condition stays true for the new variables:

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0. \quad (3.30)$$

The momentum equation in the x -direction becomes

$$\frac{\partial \phi \tilde{u}}{\partial \tilde{t}} + \frac{\partial \phi \tilde{u}^2}{\partial \tilde{x}} + \frac{\partial \phi \tilde{u} \tilde{w}}{\partial \tilde{z}} = \frac{\phi \sin \theta}{\varepsilon F^2} - \frac{1}{F^2} \frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\varepsilon}{F^2} \frac{\partial \tilde{\tau}_{xx}}{\partial \tilde{x}} + \frac{1}{\varepsilon F^2} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}}. \quad (3.31)$$

3. Modèle consistant d'écoulement granulaire incompressible – 3. Asymptotic expansion up to the first order

In the z -direction, we obtain

$$\varepsilon^2 \left(\frac{\partial \phi \tilde{w}}{\partial \tilde{t}} + \frac{\partial \phi \tilde{u} \tilde{w}}{\partial \tilde{x}} + \frac{\partial \phi \tilde{w}^2}{\partial \tilde{z}} \right) = -\frac{\phi \cos \theta}{F^2} - \frac{1}{F^2} \frac{\partial \tilde{p}}{\partial \tilde{z}} + \frac{\varepsilon}{F^2} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{x}} + \frac{\varepsilon}{F^2} \frac{\partial \tilde{\tau}_{zz}}{\partial \tilde{z}}. \quad (3.32)$$

The energy equation (3.13) gives

$$\frac{\partial \tilde{E}}{\partial \tilde{t}} + \operatorname{div} \left[\left(\tilde{E} + \frac{\tilde{p}}{F^2} \right) \tilde{\mathbf{v}} \right] = \frac{1}{F^2} \left[\tilde{u} \left(\varepsilon \frac{\partial \tilde{\tau}_{xx}}{\partial \tilde{x}} + \frac{1}{\varepsilon} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} \right) + \varepsilon \tilde{w} \left(\frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{x}} + \frac{\partial \tilde{\tau}_{zz}}{\partial \tilde{z}} \right) \right], \quad (3.33)$$

where we denoted by $\tilde{\mathbf{v}}$ the dimensionless velocity field.

Finally, we can rewrite the boundary conditions. For the velocity, we obtain

$$\tilde{u}(0) = 0, \quad \tilde{w}(0) = 0, \quad \frac{\partial h}{\partial t} + \tilde{u}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}} = \tilde{w}(\tilde{h}). \quad (3.34)$$

Equations (3.19) become in the new variables

$$\left\{ \begin{array}{l} \tilde{\tau}_{xz}(\tilde{h}) + \varepsilon \tilde{p}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}} - \varepsilon^2 \tilde{\tau}_{xx}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}} = 0, \end{array} \right. \quad (3.35a)$$

$$\left\{ \begin{array}{l} \varepsilon \tilde{\tau}_{zz}(\tilde{h}) - \tilde{p}(\tilde{h}) - \varepsilon \tilde{\tau}_{xz}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}} = 0. \end{array} \right. \quad (3.35b)$$

3. Asymptotic expansion up to the first order

We now make the hypothesis that every variable f of the problem admits an asymptotic expansion as

$$f = f^{(0)} + \varepsilon f^{(1)} + O(\varepsilon^2). \quad (3.36)$$

We thus write such an expansion for every dimensionless variable of the problem. Substituting these expansions into the mass conservation law (3.30), the momentum balance equations (3.31)–(3.32) and the constitutive law (3.26), taking into account the boundary conditions, we can compute explicitly all terms in the expansions up to order 1.

3.1. Order 0

At leading order, we obtain from Equations (3.31), (3.32), (3.35) the following relations:

$$0 = \frac{\phi \sin \theta}{F^2} + \frac{1}{F^2} \frac{\partial \tilde{\tau}_{xz}^{(0)}}{\partial \tilde{z}}, \quad (3.37)$$

$$0 = -\frac{\phi \cos \theta}{F^2} - \frac{1}{F^2} \frac{\partial \tilde{p}^{(0)}}{\partial \tilde{z}} \quad (3.38)$$

$$0 = \tilde{\tau}_{xz}^{(0)}(\tilde{h}), \quad (3.39)$$

$$0 = \tilde{p}^{(0)}(\tilde{h}). \quad (3.40)$$

We integrate the first two equations between $\tilde{z}$ and $\tilde{h}$ using the conditions at $\tilde{z} = \tilde{h}$ to obtain

$$\tilde{\tau}_{xz}^{(0)} = \phi \sin \theta (\tilde{h} - \tilde{z}) \quad (3.41)$$

and

$$\tilde{p}^{(0)} = \phi \cos \theta (\tilde{h} - \tilde{z}). \quad (3.42)$$

The pressure is thus hydrostatic at this order. Equation (3.28) gives, after using the physical assumption that $\partial \tilde{u}^{(0)} / \partial \tilde{z} > 0$,

$$|\tilde{\gamma}|^{(0)} = \frac{\partial \tilde{u}^{(0)}}{\partial \tilde{z}}. \quad (3.43)$$

Equation (3.26) then gives

$$\tilde{\tau}_{xz}^{(0)} = \mu^{(0)} \tilde{p}^{(0)} \quad (3.44)$$

with $\mu^{(0)} = \mu(I^{(0)})$. We have $\mu^{(0)} = \tau_{xz}^{(0)} / p^{(0)}$. This yields

$$\mu^{(0)} = \tan \theta. \quad (3.45)$$

At order 0, the friction coefficient has the same expression as in the case of a steady uniform flow. The expression of the inertial number at order 0 is

$$I^{(0)} = \frac{\tan \theta - \mu_1}{\mu_2 - \tan \theta} I_m \quad (3.46)$$

by (3.9). It follows from (3.43) and (3.27) that

$$\frac{\partial \tilde{u}^{(0)}}{\partial \tilde{z}} = \frac{I^{(0)}}{\tilde{d}F} \sqrt{\phi \cos \theta (\tilde{h} - \tilde{z})}, \quad (3.47)$$

that we integrate using (3.34) to obtain

$$\tilde{u}^{(0)} = \frac{2\tilde{A}}{3} \left[\tilde{h}^{3/2} - (\tilde{h} - \tilde{z})^{3/2} \right], \quad (3.48)$$

3. *Modèle consistant d'écoulement granulaire incompressible – 3. Asymptotic expansion up to the first order*

where we defined

$$\tilde{A} = \frac{I^{(0)}}{\tilde{d}F} \sqrt{\phi \cos \theta}, \quad \text{and} \quad A = \frac{I^{(0)} \sqrt{\phi g \cos \theta}}{d}. \quad (3.49)$$

At order 0, the velocity profile in the z -direction is a Bagnold profile, as expected from the $\mu(I)$ -rheology. At this order, the depth-averaged velocity is $\tilde{U}^{(0)} = 2\tilde{A}\tilde{h}^{3/2}/5$. If the characteristic velocity u_0 and the vertical scale h_0 are taken equal respectively to the depth-averaged velocity and depth of the steady uniform flow, then $\tilde{A} = 5/2$.

We deduce, using (3.30),

$$\frac{\partial \tilde{w}^{(0)}}{\partial \tilde{z}} = \tilde{A} \left[(\tilde{h} - \tilde{z})^{1/2} - \tilde{h}^{1/2} \right] \frac{\partial \tilde{h}}{\partial \tilde{x}}, \quad (3.50)$$

which is integrated as

$$\tilde{w}^{(0)} = \tilde{A} \left[\frac{2}{3} \tilde{h}^{3/2} - \frac{2}{3} (\tilde{h} - \tilde{z})^{3/2} - \tilde{z} \tilde{h}^{1/2} \right] \frac{\partial \tilde{h}}{\partial \tilde{x}}. \quad (3.51)$$

Finally, we use (3.26) to obtain

$$\tilde{\tau}_{xx}^{(0)} = -\tilde{\tau}_{zz}^{(0)} = 2\phi \sin \theta \left[\tilde{h}^{1/2} (\tilde{h} - \tilde{z})^{1/2} - \tilde{h} + \tilde{z} \right] \frac{\partial \tilde{h}}{\partial \tilde{x}}. \quad (3.52)$$

3.2. Order 1

We now investigate Equations (3.31), (3.32) and (3.35) at the first order. At this order of accuracy, these equations become

$$\frac{\partial \tilde{u}^{(0)}}{\partial \tilde{t}} + \tilde{u}^{(0)} \frac{\partial \tilde{u}^{(0)}}{\partial \tilde{x}} + \tilde{w}^{(0)} \frac{\partial \tilde{u}^{(0)}}{\partial \tilde{z}} = -\frac{1}{\phi F^2} \frac{\partial \tilde{p}^{(0)}}{\partial \tilde{x}} + \frac{1}{\phi F^2} \frac{\partial \tilde{\tau}_{xz}^{(1)}}{\partial \tilde{z}}, \quad (3.53)$$

$$0 = -\frac{1}{F^2} \frac{\partial \tilde{p}^{(1)}}{\partial \tilde{z}} + \frac{1}{F^2} \frac{\partial \tilde{\tau}_{xz}^{(0)}}{\partial \tilde{x}} + \frac{1}{F^2} \frac{\partial \tilde{\tau}_{zz}^{(0)}}{\partial \tilde{z}} \quad (3.54)$$

$$0 = \tilde{\tau}_{xz}^{(1)}(\tilde{h}) + \tilde{p}^{(0)}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}}, \quad (3.55)$$

$$0 = \tilde{\tau}_{zz}^{(0)}(\tilde{h}) - \tilde{p}^{(1)}(\tilde{h}) - \tilde{\tau}_{xz}^{(0)}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}}. \quad (3.56)$$

In view of the values of $\tilde{\tau}_{xz}^{(0)}(\tilde{h})$, $\tilde{\tau}_{zz}^{(0)}(\tilde{h})$ and $\tilde{p}^{(0)}(\tilde{h})$ (see (3.41), (3.52) and (3.42)), we obtain from the two last equations

$$\tilde{\tau}_{xz}^{(1)}(\tilde{h}) = \tilde{p}^{(1)}(\tilde{h}) = 0. \quad (3.57)$$

3. Modèle consistant d'écoulement granulaire incompressible – 3. Asymptotic expansion up to the first order

We substitute the values of $\tilde{u}^{(0)}$, $\tilde{w}^{(0)}$ and $\tilde{p}^{(0)}$, using Equation (3.34) to express $\partial\tilde{h}/\partial\tilde{t}$ at the leading order as

$$\frac{\partial\tilde{h}}{\partial\tilde{t}} = -\tilde{A}\tilde{h}^{3/2}\frac{\partial\tilde{h}}{\partial\tilde{x}} + O(\varepsilon). \quad (3.58)$$

Equation (3.58) is the kinematic wave equation for granular flows. This equation was written by Börzsönyi, Halsey, et al. 2005 at the small Froude number limit where there is an additional diffusive term due to the pressure. In the present case, with $F = O(1)$, the kinematic wave equation is obtained at leading order without diffusive term. The kinematic wave velocity is $\tilde{a}_0 = \tilde{A}\tilde{h}^{3/2}$. It is the same value as found by Börzsönyi, Halsey, et al. 2005 with the expression of I_m of Jop et al. 2005, Appendix A.

We obtain

$$\frac{\partial\tilde{\tau}_{xz}^{(1)}}{\partial\tilde{z}} = \phi\frac{\partial\tilde{h}}{\partial\tilde{x}} \left[\cos\theta + \frac{F^2\tilde{A}^2}{3} (\tilde{h}^{1/2}(\tilde{h}-\tilde{z})^{3/2} - \tilde{h}^2) \right]. \quad (3.59)$$

This expression is integrated between $\tilde{z}$ and $\tilde{h}$ with the boundary condition. This yields

$$\tilde{\tau}_{xz}^{(1)} = \phi\frac{\partial\tilde{h}}{\partial\tilde{x}} \left[\frac{F^2\tilde{A}^2}{3} \left((\tilde{h}-\tilde{z})\tilde{h}^2 - \frac{2}{5}\tilde{h}^{1/2}(\tilde{h}-\tilde{z})^{5/2} \right) - \cos\theta(\tilde{h}-\tilde{z}) \right]. \quad (3.60)$$

We now substitute the values of $\tilde{\tau}_{xz}^{(0)}$ and $\tilde{\tau}_{zz}^{(0)}$ into the second equation. We find

$$\frac{\partial\tilde{p}^{(1)}}{\partial\tilde{z}} = \phi\sin\theta\frac{\partial\tilde{h}}{\partial\tilde{x}} \left[\frac{\tilde{h}^{1/2}}{(\tilde{h}-\tilde{z})^{1/2}} - 1 \right], \quad (3.61)$$

which is integrated between $\tilde{z}$ and $\tilde{h}$ to calculate the expression of the first-order correction to the pressure as

$$\tilde{p}^{(1)} = \phi\sin\theta\frac{\partial\tilde{h}}{\partial\tilde{x}} \left[\tilde{h}-\tilde{z} - 2\tilde{h}^{1/2}(\tilde{h}-\tilde{z})^{1/2} \right]. \quad (3.62)$$

We also obtain from (3.28)

$$|\tilde{\gamma}|^{(1)} = \frac{\partial\tilde{u}^{(1)}}{\partial\tilde{z}}. \quad (3.63)$$

Hence, with (3.26), we can write $\tilde{\tau}_{xz}^{(1)} = \mu(I)^{(1)}\tilde{p}^{(0)} + \mu(I)^{(0)}\tilde{p}^{(1)}$. Therefore, the first-order correction to the friction coefficient can be consistently written as $\mu^{(1)} = (\tilde{\tau}_{xz}^{(1)} - \mu^{(0)}\tilde{p}^{(1)})/\tilde{p}^{(0)}$, which leads to the expression

$$\mu^{(1)} = \frac{\partial\tilde{h}}{\partial\tilde{x}} \left[\frac{F^2\tilde{A}^2}{3\cos\theta} \left(\tilde{h}^2 - \frac{2}{5}\tilde{h}^{1/2}(\tilde{h}-\tilde{z})^{3/2} \right) - \frac{1}{\cos^2\theta} + 2\tilde{h}^{1/2}(\tilde{h}-\tilde{z})^{-1/2}\tan^2\theta \right]. \quad (3.64)$$

We now deduce the expression of the first-order correction to the inertial number, writing

$$I^{(1)} = \frac{\partial I}{\partial\mu} \Big|_{\mu^{(0)}} \mu^{(1)}, \quad (3.65)$$

3. Modèle consistant d'écoulement granulaire incompressible – 4. Derivation of the depth-averaged equations

to find

$$I^{(1)} = I_m \frac{\mu_2 - \mu_1}{(\mu_2 - \tan \theta)^2} \mu^{(1)}. \quad (3.66)$$

We can write

$$I^{(1)} = B I^{(0)} \mu^{(1)} \quad (3.67)$$

where $B = (\partial \ln I / \partial \mu)(\mu = \tan \theta)$ or

$$B = \frac{\mu_2 - \mu_1}{(\mu_2 - \tan \theta)(\tan \theta - \mu_1)}. \quad (3.68)$$

At first order, the expansion of (3.27) gives

$$|\tilde{\gamma}|^{(1)} = \frac{1}{\tilde{d}F} \left(I^{(1)} \sqrt{p^{(0)}} + \frac{I^{(0)} p^{(1)}}{2\sqrt{p^{(0)}}} \right). \quad (3.69)$$

Equations (3.62), (3.67), (3.63) and (3.42) can be used to obtain

$$\begin{aligned} \frac{\partial \tilde{u}^{(1)}}{\partial \tilde{z}} = \tilde{A} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left\{ \frac{BF^2 \tilde{A}^2}{3 \cos \theta} \left[\tilde{h}^2 (\tilde{h} - \tilde{z})^{1/2} - \frac{2}{5} \tilde{h}^{1/2} (\tilde{h} - \tilde{z})^2 \right] - \frac{B (\tilde{h} - \tilde{z})^{1/2}}{\cos^2 \theta} \right. \\ \left. + 2B \tilde{h}^{1/2} \tan^2 \theta + \frac{\tan \theta}{2} (\tilde{h} - \tilde{z})^{1/2} - \tilde{h}^{1/2} \tan \theta \right\}. \end{aligned} \quad (3.70)$$

The integration of this expression leads to

$$\begin{aligned} u^{(1)} = \tilde{A} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left\{ \frac{2BF^2 \tilde{A}^2}{9 \cos \theta} \left[\tilde{h}^2 \left(\tilde{h}^{3/2} - (\tilde{h} - \tilde{z})^{3/2} \right) - \frac{\tilde{h}^{1/2}}{5} \left(\tilde{h}^3 - (\tilde{h} - \tilde{z})^3 \right) \right] \right. \\ \left. + \left(\tan \theta - \frac{2B}{\cos^2 \theta} \right) \frac{\tilde{h}^{3/2} - (\tilde{h} - \tilde{z})^{3/2}}{3} + (2B \tan \theta - 1) \tilde{h}^{1/2} \tilde{z} \tan \theta \right\}, \end{aligned} \quad (3.71)$$

which is the consistent first-order correction to the Bagnold velocity profile.

4. Derivation of the depth-averaged equations

4.1. Averaged variables

We now define depth-averaged variables, and give an explicit asymptotic expansion of these variables up to the first order. The average over the depth of any quantity X is defined as

$$\langle X \rangle = \frac{1}{h} \int_0^h X \, dz. \quad (3.72)$$

3. Modèle consistant d'écoulement granulaire incompressible – 4. Derivation of the depth-averaged equations

To lighten the notation, the depth-averaged velocity is denoted by $U = \langle u \rangle$. The average velocity $\tilde{U}$ admits an asymptotic expansion given by

$$\tilde{U} = \tilde{U}^{(0)} + \varepsilon \tilde{U}^{(1)} + O(\varepsilon^2), \quad (3.73)$$

where, by (3.48),

$$\tilde{U}^{(0)} = \frac{2}{5} \tilde{A} \tilde{h}^{3/2}, \quad (3.74)$$

and, from (3.71), after some computations,

$$\tilde{U}^{(1)} = \frac{\tilde{A}B}{10} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\frac{F^2 \tilde{A}^2}{\cos \theta} \tilde{h}^{7/2} + \left(6 \tan^2 \theta - 4 - \frac{3 \tan \theta}{B} \right) \tilde{h}^{3/2} \right]. \quad (3.75)$$

Note that the first-order correction $U^{(1)}$ can be written

$$\tilde{U}^{(1)} = \frac{\tilde{U} - \tilde{U}^{(0)}}{\varepsilon} + O(\varepsilon) = \frac{1}{\varepsilon} \left(\tilde{U} - \frac{2}{5} \tilde{A} \tilde{h}^{3/2} \right) + O(\varepsilon), \quad (3.76)$$

To simplify the notations, we define $C = \tan \theta (2 \tan \theta - 1/B)$, i.e.

$$C = \tan^2 \theta + \tan \theta \left[\frac{(\tan \theta - \mu_1)^2}{\mu_2 - \mu_1} + \mu_1 \right]. \quad (3.77)$$

Note that $C = C(\theta)$ is a positive increasing function of θ . It takes values in the interval $[2\mu_1^2, 2\mu_2^2]$ if $\theta \in [\arctan \mu_1, \arctan \mu_2]$. With this new notation, the function $\tilde{U}^{(1)}$ can be written:

$$\tilde{U}^{(1)} = \frac{\tilde{A}B}{10} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\frac{F^2 \tilde{A}^2}{\cos \theta} \tilde{h}^{7/2} + (3C - 4) \tilde{h}^{3/2} \right]. \quad (3.78)$$

We also define a quantity, called enstrophy, as $\psi = \langle (u - U)^2 \rangle / h^2$. The enstrophy takes into account shear effects. It is zero for a velocity constant in the depth and positive otherwise. The enstrophy is scaled as $\psi = \tilde{\psi} u_0^2 / h_0^2$. With this definition, $\tilde{h} \langle \tilde{u}^2 \rangle = \tilde{h} \tilde{U}^2 + \tilde{h}^3 \tilde{\psi}$. Similarly, $\tilde{\psi}$ expands as $\tilde{\psi} = \tilde{\psi}^{(0)} + \varepsilon \tilde{\psi}^{(1)} + O(\varepsilon^2)$, where

$$\tilde{\psi}^{(0)} = \frac{\tilde{A}^2 \tilde{h}}{25} \quad (3.79)$$

is the consistent expression of the enstrophy for a Bagnold profile, and

$$\tilde{\psi}^{(1)} = \frac{\tilde{A}^2 B}{25} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\frac{6F^2 \tilde{A}^2}{11 \cos \theta} \tilde{h}^3 + \left(\frac{13}{7} C - 2 \right) \tilde{h} \right]. \quad (3.80)$$

At leading order, the enstrophy and the depth-averaged velocity satisfy the relation

$$\tilde{\psi}^{(0)} = \frac{(\tilde{U}^{(0)})^2}{4 \tilde{h}^2}. \quad (3.81)$$

Note that this expression gives at leading order $h \langle (u^{(0)})^2 \rangle = (5/4) h (U^{(0)})^2$. This is

3. Modèle consistant d'écoulement granulaire incompressible – 4. Derivation of the depth-averaged equations

consistent with the shape factor $5/4$ which is obtained for a Bagnold velocity profile. This value was confirmed by the experimental measurements of Saingier et al. 2016. However, using the additional quantity enstrophy and a three-equation model instead of a shape factor and a two-equation model modifies and improves the mathematical structure of the model. In particular, a three-equation model admits an energy balance equation, which is not the case for a two-equation model with a shape factor different from 1 (Richard and Gavriluk 2012, Appendix A).

The relation (3.81) leads to the expansion

$$\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{h}^2} = \varepsilon \left(\tilde{\psi}^{(1)} - \frac{\tilde{A}\tilde{U}^{(1)}}{5\tilde{h}^{1/2}} \right) + O(\varepsilon^2). \quad (3.82)$$

The leading term of (3.82) can be computed with Equations (3.80) and (3.78). We obtain

$$\tilde{\psi}^{(1)} - \frac{\tilde{A}\tilde{U}^{(1)}}{5\tilde{h}^{1/2}} = \frac{\tilde{A}^2 B}{10} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{F^2 \tilde{A}^2}{55 \cos \theta} \tilde{h}^3 + \frac{C}{7} \tilde{h} \right). \quad (3.83)$$

This expression is useful to write consistently a relaxation term for the enstrophy.

4.2. Depth-averaged mass and momentum equations

We now want to derive evolution equations for the depth-averaged variables. We first integrate the divergence free condition (3.30) between 0 and $\tilde{h}$. Using the kinematic boundary condition (3.34), we obtain the depth-averaged mass conservation equation

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{x}} = 0. \quad (3.84)$$

As in Börzsönyi, Halsey, et al. 2005, using this mass conservation equation with the zeroth-order expression (3.74) of the average velocity gives the kinematic wave equation (3.58) to within first-order terms, except for the diffusive term of Börzsönyi, Halsey, et al. 2005, which is of order 1 in this asymptotic.

In order to derive an equation for $\tilde{U}$, we integrate (3.31) between 0 and $\tilde{h}$ using the boundary condition (3.34). This leads to a preliminary expression of the depth-averaged momentum balance equation

$$\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\tilde{h}\tilde{U}^2 + \tilde{h}^3 \tilde{\psi} + \frac{1}{\phi F^2} \int_0^{\tilde{h}} \tilde{p} d\tilde{z} \right) = \frac{1}{\varepsilon \phi F^2} (\phi \sin \theta \tilde{h} - \tilde{\tau}_{xz}(0)) + O(\varepsilon). \quad (3.85)$$

We first compute the integral of the pressure $\tilde{p}$. By using the expression (3.42) of $\tilde{p}^{(0)}$, we obtain

$$\int_0^{\tilde{h}} \tilde{p} d\tilde{z} = \int_0^{\tilde{h}} \tilde{p}^{(0)} d\tilde{z} + O(\varepsilon) = \frac{\phi \tilde{h}^2 \cos \theta}{2} + O(\varepsilon). \quad (3.86)$$

3. Modèle consistant d'écoulement granulaire incompressible – 4. Derivation of the depth-averaged equations

Equation (3.85) thus yields

$$\frac{\partial \tilde{h} \tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\tilde{h} \tilde{U}^2 + \tilde{h}^3 \tilde{\psi} + \frac{\tilde{h}^2 \cos \theta}{2F^2} \right) = \frac{1}{\varepsilon \phi F^2} (\phi \tilde{h} \sin \theta - \tilde{\tau}_{xz}(0)) + O(\varepsilon). \quad (3.87)$$

We now need to find an expression for the basal friction $\tilde{\tau}_{xz}(0)$. The basal friction for a granular flow can be written $\tilde{\tau}_{xz}(0) = \mu_b \phi \tilde{h} \cos \theta$ (Andreotti et al. 2013) where

$$\mu_b = \mu_1 + \frac{\mu_2 - \mu_1}{I_m \frac{2\tilde{h}^{3/2} \sqrt{\phi \cos \theta}}{5\tilde{d}F\tilde{U}} + 1}. \quad (3.88)$$

This expression is found assuming a Bagnold velocity profile i.e. a steady uniform flow. In the framework of the asymptotic method, it means that this expression is valid at order zero. However the velocity profile is modified for variable flows, in particular due to the effects of the inertial terms. It follows that the first-order correction to this expression must be calculated in order to derive a first-order consistent model. This expression can be written

$$\mu_b = \mu_1 + \frac{\mu_2 - \mu_1}{\frac{I_m}{I^{(0)}} \frac{\tilde{U}^{(0)}}{\tilde{U}} + 1} \quad (3.89)$$

which means that $\mu_b = \mu(I = I^{(0)} \tilde{U} / \tilde{U}^{(0)})$ such that $\mu_b^{(0)} = \mu^{(0)}$. This friction law is thus consistent at order 0 with the $\mu(I)$ rheology, and it was used in depth-averaged models (Jop et al. 2005; Forterre 2006; Gray and Edwards 2014). However, whereas $\tilde{\tau}_{xz}^{(0)}(0) = \mu_b^{(0)} \phi \tilde{h} \cos \theta$, it is important to stress that this relation is not satisfied at order 1 (i.e. $\tilde{\tau}_{xz}^{(1)}(0) \neq \mu_b^{(1)} \phi \tilde{h} \cos \theta$ where μ_b is given by (3.88)), which implies that the basal friction law (3.88) or (3.89) is not consistent at order 1 with the rheology.

The strategy for obtaining a consistent model is to retain Expression (3.89) by adding first-order corrective terms, written in the form of a relaxation term or as corrective factors in the flux terms. In this way, given that the relaxation and flux terms cancel out by definition at equilibrium, the friction law (3.89) will be found for steady uniform flows, for which it is valid. The method used to achieve this result is to expand the expression (3.89) in powers of ε using the expansion of $\tilde{U} = \tilde{U}^{(0)} + \varepsilon \tilde{U}^{(1)} + O(\varepsilon^2)$. The expansion of $\mu_b = \mu(I = I^{(0)} \tilde{U} / \tilde{U}^{(0)})$ gives

$$\mu_b = \mu^{(0)} + \varepsilon \left. \frac{\partial \mu}{\partial I} \right|_{I^{(0)}} \frac{I^{(0)}}{\tilde{U}^{(0)}} \tilde{U}^{(1)} + O(\varepsilon^2) \quad (3.90)$$

which can be written as

$$\mu_b = \mu^{(0)} + \varepsilon \frac{\tilde{U}^{(1)}}{B \tilde{U}^{(0)}} + O(\varepsilon^2). \quad (3.91)$$

Using Equations (3.41), (3.42) and (3.60), the shear stress at the bottom can be written

$$\tilde{\tau}_{xz}(0) = \mu^{(0)} \phi \tilde{h} \cos \theta + \varepsilon \phi \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{F^2 \tilde{A}^2}{5} \tilde{h}^3 - \tilde{h} \cos \theta \right) + O(\varepsilon^2). \quad (3.92)$$

3. Modèle consistant d'écoulement granulaire incompressible – 4. Derivation of the depth-averaged equations

Subtracting $\mu_b \phi \tilde{h} \cos \theta$ using (3.91) from $\tilde{\tau}_{xz}(0)$ leads to the expression of the first-order correction to the friction law

$$\tilde{\tau}_{xz}(0) - \mu_b \phi \tilde{h} \cos \theta = \varepsilon \phi R \cos \theta + O(\varepsilon^2), \quad (3.93)$$

where

$$R = -\frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{F^2 \tilde{A}^2}{20 \cos \theta} \tilde{h}^3 + \frac{3C}{4} \tilde{h} \right). \quad (3.94)$$

It follows that the difference $\tilde{h} \sin \theta - \tilde{\tau}_{xz}(0)/\phi$ cannot be written simply $\tilde{h} \cos \theta (\tan \theta - \mu_b) + O(\varepsilon^2)$. This difference can be written $\tilde{h} \cos \theta (\tan \theta - \mu_b) - \varepsilon R \cos \theta + O(\varepsilon^2)$. The difference $\tilde{h} \cos \theta (\tan \theta - \mu_b)$, which is of $O(\varepsilon)$ even if ε does not appear in the expression, is called a relaxation term. It is equal to zero for a steady uniform flow and determines the properties of these equilibrium flows. For a variable flow, this difference is small (of $O(\varepsilon)$) and the corrective term in R does not vanish.

Equation (3.87) can then be written

$$\frac{\partial \tilde{h} \tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\tilde{h} \tilde{U}^2 + \tilde{h}^3 \tilde{\psi} + \frac{\tilde{h}^2 \cos \theta}{2F^2} \right) = \frac{\tilde{h} \cos \theta}{\varepsilon F^2} (\tan \theta - \mu_b) - R \frac{\cos \theta}{F^2} + O(\varepsilon). \quad (3.95)$$

In order to derive a model with a satisfactory mathematical structure (i.e. a hyperbolic model in conservative form with relaxation source terms), we now need to find an expression for the first-order correction to the friction law, i.e. the term $-R \cos \theta / F^2$ with its problematic derivative $\partial \tilde{h} / \partial \tilde{x}$. We choose to split R into two terms. The first term is written in the left-hand side of Equation (3.95) as a flux, and the second term is written in the right-hand side and is expressed as a relaxation term. More precisely, for any $\lambda \geq 0$, we can write consistently, using Equation (3.83)

$$\begin{aligned} -\frac{\cos \theta}{F^2} R = C \left(\frac{3}{4} - \frac{\lambda}{7} \right) \frac{\partial}{\partial \tilde{x}} \left(\frac{\tilde{h}^2 \cos \theta}{2F^2} \right) + \left(\frac{5}{16} - \frac{5\lambda}{44} \right) \frac{\partial}{\partial \tilde{x}} (\tilde{h}^3 \tilde{\psi}^{(0)}) \\ + \lambda \frac{10 \cos \theta}{F^2 \tilde{A}^2 B} \left(\tilde{\psi}^{(1)} - \frac{\tilde{A} \tilde{U}^{(1)}}{5 \tilde{h}^{1/2}} \right). \end{aligned} \quad (3.96)$$

The parameter λ can be freely chosen with no loss of consistency. There is thus an infinite number of consistent models, which are all equivalent in the validity domain of the asymptotic expansion, i.e. in the shallow-water approximation, but which differ outside this domain. The final choice (3.134) of the expression of λ is explained in §5.3 with validations in §5.4.

We can now substitute this expression of R into Equation (3.95). We obtain

$$\frac{\partial \tilde{h} \tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\tilde{h} \tilde{U}^2 + \beta \tilde{h}^3 \tilde{\psi} + K \frac{\tilde{h}^2 \cos \theta}{2F^2} \right) = \frac{\cos \theta}{\varepsilon F^2} \left[\tilde{h} (\tan \theta - \mu_b) + \frac{10\lambda}{\tilde{A}^2 B} \left(\tilde{\psi} - \frac{\tilde{U}^2}{4 \tilde{h}^2} \right) \right] + O(\varepsilon), \quad (3.97)$$

3. Modèle consistant d'écoulement granulaire incompressible – 4. Derivation of the depth-averaged equations

where

$$\beta = \frac{11}{16} + \frac{5\lambda}{44}, \quad \text{and} \quad K = 1 - \frac{3C}{4} + \frac{\lambda C}{7}. \quad (3.98)$$

Equation (3.97) is consistent at the first order of accuracy. The first-order correction to the usual granular friction law is written partly as a relaxation term in the right-hand side and partly in the momentum flux as corrections to the hydrostatic pressure term (K) and to the enstrophy term (β). The latter term also models shearing effects in the granular flow. The structure of this equation is similar to other models with enstrophy (Denisenko, Richard, et al. 2023 for example) with an effective enstrophy equal to $\beta\psi$. There is also a factor K in the hydrostatic pressure term.

4.3. Depth-averaged energy equation

In order to close the system of equations obtained for $\tilde{h}$ and $\tilde{U}$ in the previous section, we need to find an evolution equation for the variable $\tilde{\psi}$. Such an equation can be obtained from the equation of energy (3.33). We first rewrite equation (3.33) as:

$$\frac{1}{2} \frac{\partial \tilde{u}^2}{\partial \tilde{t}} + \text{div} \left[\left(\frac{\tilde{u}^2}{2} - \frac{\tilde{x} \sin \theta}{\varepsilon F^2} + \frac{\phi \tilde{z} \cos \theta + \tilde{p}}{\phi F^2} \right) \tilde{\mathbf{v}} \right] = \frac{1}{\varepsilon \phi F^2} \tilde{u} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} + O(\varepsilon). \quad (3.99)$$

The integration of Equation (3.99) between 0 and $\tilde{h}$, using the boundary conditions (3.34) and (3.35b) and the incompressibility condition (3.30), gives

$$\begin{aligned} & \frac{\partial}{\partial \tilde{t}} \left(\int_0^{\tilde{h}} \frac{\tilde{u}^2}{2} d\tilde{z} + \frac{\tilde{h}^2}{2F^2} \cos \theta \right) + \frac{\partial}{\partial \tilde{x}} \int_0^{\tilde{h}} \left(\frac{\tilde{u}^3}{2} + \frac{\tilde{u}(\tilde{p} + \phi \tilde{z} \cos \theta)}{\phi F^2} \right) d\tilde{z} - \frac{\tilde{h} \tilde{U}}{\varepsilon F^2} \sin \theta \\ &= \frac{1}{\varepsilon \phi F^2} \int_0^{\tilde{h}} \tilde{u} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} d\tilde{z} + O(\varepsilon). \end{aligned} \quad (3.100)$$

As in the previous section, we have $h\langle u^2 \rangle = hU^2 + h^3\psi$. Similarly, $h\langle u^3 \rangle$ can be written as $h\langle u^3 \rangle = hU^3 + 3Uh^3\psi + h\langle (u-U)^3 \rangle$. In order to close the equations, we need a formula for the third-order correlation $\langle (u-U)^3 \rangle$. Since we neglect second-order terms, it is consistent to replace this term by the leading term in the asymptotic expansion, which is $h\langle (u-U)^3 \rangle^{(0)} = -4\tilde{A}^3\tilde{h}^{11/2}/1375$. Hence

$$\frac{\partial}{\partial \tilde{x}} \int_0^{\tilde{h}} \frac{1}{2} (\tilde{u}^{(0)} - \tilde{U}^{(0)})^3 d\tilde{z} = -\frac{\tilde{A}^3\tilde{h}^{9/2}}{125} \frac{\partial \tilde{h}}{\partial \tilde{x}}. \quad (3.101)$$

This term is written in the right-hand side of the equation in order to obtain a model with a well-posed and simple mathematical structure. Equation (3.42) shows that, at order 0, we can write

$$\tilde{p}^{(0)} + \phi \tilde{z} \cos \theta = \phi \tilde{h} \cos \theta. \quad (3.102)$$

3. Modèle consistant d'écoulement granulaire incompressible – 4. Derivation of the depth-averaged equations

We can use this relation to write

$$\int_0^{\tilde{h}} \frac{\tilde{u}(\tilde{p} + \tilde{z} \cos \theta)}{\phi F^2} d\tilde{z} = \frac{\tilde{h}^2 \tilde{U} \cos \theta}{F^2} + O(\varepsilon). \quad (3.103)$$

The technical details of the treatment of the right-hand side of Equation (3.100) are given in Appendix 8.1. The calculations are carried out to ensure compatibility between the energy and momentum depth-averaged equations in order to obtain a well-posed mathematical structure well suited to numerical resolution.

The final version of the energy equation can be written

$$\begin{aligned} & \frac{\partial}{\partial \tilde{t}} \left(\tilde{h} \frac{\tilde{U}^2}{2} + \frac{\beta \tilde{h}^3 \tilde{\psi}}{2} + K \frac{\tilde{h}^2 \cos \theta}{2F^2} \right) + \frac{\partial}{\partial \tilde{x}} \left(\frac{\tilde{h} \tilde{U}^3}{2} + \frac{3\beta \tilde{h}^3 \tilde{U} \tilde{\psi}}{2} + K \frac{\tilde{h}^2 \tilde{U} \cos \theta}{F^2} \right) \\ &= \frac{\tilde{U} \cos \theta}{\varepsilon F^2} \left[\tilde{h}(\tan \theta - \mu_b) + \frac{\beta \tilde{\alpha}_1}{\tilde{h}^{1/2}} \left(\tilde{U} - \frac{2}{5} \tilde{A} \tilde{h}^{3/2} \right) + \left(\beta \tilde{\alpha}_2 + \frac{10\lambda}{\tilde{A}^2 B} \right) \left(\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{h}^2} \right) \right] + O(\varepsilon) \end{aligned} \quad (3.104)$$

with

$$\tilde{\alpha}_1 = -\frac{33C}{2\tilde{A}B(34C+28)}, \quad (3.105)$$

and

$$\tilde{\alpha}_2 = \frac{77(9C-12)}{2\tilde{A}^2 B(34C+28)}. \quad (3.106)$$

As in the momentum equation, the effective enstrophy in this energy equation is $\beta\psi$ and there is the same factor K in the pressure term and in the potential energy. The right-hand side of this energy equation includes the power of the weight and of the usual granular friction law (3.88) but there are additional relaxation terms to obtain consistency at order 1.

The evolution equation for the variable $\tilde{\psi}$ can now be derived. In order to do this, the equation of momentum (3.97) is multiplied by $\tilde{U}$ and this new equation is subtracted from the equation of energy (3.104). This leads to

$$\frac{\partial \tilde{h} \tilde{\psi}}{\partial \tilde{t}} + \frac{\partial \tilde{h} \tilde{U} \tilde{\psi}}{\partial \tilde{x}} = \frac{2\tilde{U} \cos \theta}{\varepsilon \tilde{h}^2 F^2} \left[\frac{\tilde{\alpha}_1}{\tilde{h}^{1/2}} \left(\tilde{U} - \frac{2}{5} \tilde{A} \tilde{h}^{3/2} \right) + \tilde{\alpha}_2 \left(\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{h}^2} \right) \right] + O(\varepsilon). \quad (3.107)$$

The evolution equation of the enstrophy is a transport equation with relaxation source terms. The relaxation term on ψ is important and usual for an enstrophy equation and the coefficient $\tilde{\alpha}_2$ must be negative for the system to be mathematically well-posed (otherwise the enstrophy diverges exponentially). This gives the condition $C < 4/3$, which is discussed below in §6. For a slope in the range $[\mu_1, \mu_2]$, this condition is automatically satisfied if $\mu_2 < \sqrt{2/3} \approx 0.816 \approx \tan(39.2^\circ)$ (see §6).

5. Properties and extensions of the model

5.1. Reconstruction of the velocity field

Since the model is derived consistently from the asymptotic expansions of the flow variables, it is possible to reconstruct the 2D-fields and, in particular, the variations of the velocity in the depth, from the values of the depth-averaged quantities calculated with the 1D-depth-averaged model. These velocity profiles are calculated at the same accuracy as the model. In the following, we show how to reconstruct in a consistent way the horizontal velocity field from the solution of (3.116), i.e. from the triplet (h, U, ψ) .

Reverting to the dimensionless quantities, the expressions of $\tilde{u}^{(0)}$ at equilibrium (3.48) and of the averaged velocity $\tilde{U}^{(0)}$ (3.74) lead to

$$\tilde{u}^{(0)} = f(z)\tilde{U}^{(0)}, \quad (3.108)$$

where

$$f(z) = \frac{5}{3} \left[1 - \left(1 - \frac{z}{h} \right)^{3/2} \right] \quad (3.109)$$

is the renormalised Bagnold profile. From this relation, we deduce that an approximation of $\tilde{u}$ at order 0 is given by $\tilde{u} = f(z)\tilde{U} + O(\varepsilon)$. In order to have an approximation of order 1, we expand

$$\tilde{u} = f(z)\tilde{U} + \varepsilon (\tilde{u}^{(1)} - f(z)\tilde{U}^{(1)}) + O(\varepsilon^2). \quad (3.110)$$

Thus we can obtain a consistent expression for $\tilde{u}$ by expressing consistently $\tilde{u}^{(1)} - f(z)\tilde{U}^{(1)}$ as a function of $(\tilde{h}, \tilde{U}, \tilde{\psi})$. After using (3.71) and (3.78) and some computations, we obtain

$$\tilde{u}^{(1)} - f(z)\tilde{U}^{(1)} = \tilde{A}B \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\frac{F^2 \tilde{A}^2}{25 \cos \theta} \tilde{h}^{7/2} f(z) \left(\frac{2}{5} f(z) - \frac{1}{2} \right) + C \tilde{h}^{3/2} \left(\frac{\tilde{z}}{\tilde{h}} - \frac{1}{2} f(z) \right) \right], \quad (3.111)$$

which yields, using (3.78) and (3.83),

$$\tilde{u}^{(1)} - f(z)\tilde{U}^{(1)} = f_1(z)\tilde{U}^{(1)} + f_2(z) \left(\tilde{\psi}^{(1)} - \frac{\tilde{A}\tilde{U}^{(1)}}{5\tilde{h}^{1/2}} \right), \quad (3.112)$$

where

$$f_1(z) = \frac{2C}{34C + 28} \left[2f(z) \left(\frac{11}{5} f(z) + 6 \right) - 35 \frac{z}{h} \right] \quad (3.113)$$

and

$$f_2(z) = \frac{55\tilde{h}^{1/2}}{A(17C/7 + 2)} \left[5C \left(\frac{z}{h} - \frac{1}{2} f(z) \right) - \frac{1}{5} (3C - 4) f(z) \left(\frac{2}{5} f(z) - \frac{1}{2} \right) \right]. \quad (3.114)$$

As we did before, we finally use (3.76) and (3.82) to replace the terms of first order in

the right-hand side of (3.112) and, together with (3.110), we obtain, reverting to the dimensional quantities,

$$u = f(z)U + f_1(z) \left(U - \frac{2A}{5} h^{3/2} \right) + f_2(z) \left(\psi - \frac{U^2}{4h^2} \right). \quad (3.115)$$

This expression gives the velocity profile in the flow to the accuracy of the model, i.e. to within terms of order 2.

5.2. Hyperbolicity

In this section, we study some properties of the model obtained in Section 4 and formed by the mass conservation equation (3.84), the momentum balance equation (3.97) and the enstrophy equation (3.107). In dimensional form, this system reads

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0, \\ \frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left(hU^2 + \beta h^3 \psi + K \frac{gh^2}{2} \cos \theta \right) = g \cos \theta \left[h(\tan \theta - \mu_b) + \frac{10\lambda}{A^2 B} \left(\psi - \frac{U^2}{4h^2} \right) \right], \\ \frac{\partial h\psi}{\partial t} + \frac{\partial hU\psi}{\partial x} = \frac{2gU \cos \theta}{h^2} \left[\frac{\alpha_1}{h^{1/2}} \left(U - \frac{2}{5} Ah^{3/2} \right) + \alpha_2 \left(\psi - \frac{U^2}{4h^2} \right) \right], \end{cases} \quad (3.116)$$

where the expressions of A , B , C , α_1 and α_2 are

$$A = I_m \frac{\tan \theta - \mu_1}{\mu_2 - \tan \theta} \frac{\sqrt{\phi g \cos \theta}}{d}, \quad B = \frac{\mu_2 - \mu_1}{(\mu_2 - \tan \theta)(\tan \theta - \mu_1)}, \quad C = \tan \theta \left(2 \tan \theta - \frac{1}{B} \right), \quad (3.117)$$

$$\alpha_1 = -\frac{33C}{2AB(34C + 28)}, \quad \alpha_2 = \frac{77(9C - 12)}{2A^2 B(34C + 28)}. \quad (3.118)$$

The parameters λ , β and K are given by

$$\lambda = \frac{847(4 - 3C)}{20(157C + 84)}, \quad \beta = \frac{11}{16} + \frac{5\lambda}{44}, \quad K = 1 - \frac{3C}{4} + \frac{\lambda C}{7}, \quad (3.119)$$

and the basal friction μ_b is given by

$$\mu_b = \mu_1 + \frac{\mu_2 - \mu_1}{I_m \frac{2h^{3/2} \sqrt{\phi g \cos \theta}}{5dU} + 1}. \quad (3.120)$$

The three characteristic velocities are

$$\lambda_0 = U, \quad \lambda_{\pm} = U \pm \sqrt{Kgh \cos \theta + 3\beta h^2 \psi}. \quad (3.121)$$

The system of equations (3.116) is hyperbolic if $K > 0$. It is in conservative form, with relaxation source terms. For the final expression (3.134) of λ or for $\lambda = 0$, the condition

$K > 0$ is equivalent to the condition $C < 4/3$, which is also a condition for a well-posed system (see above in §4.3) and which is discussed in §6. If $K < 0$, the hyperbolicity is not guaranteed, which is not admissible.

The system admits the additional energy balance equation

$$\begin{aligned} \frac{\partial h e}{\partial t} + \frac{\partial}{\partial x} (h U e + \Pi U) \\ = g U \cos \theta \left[h(\tan \theta - \mu_b) + \frac{\beta \alpha_1}{h^{1/2}} \left(U - \frac{2}{5} A h^{3/2} \right) + \left(\beta \alpha_2 + \frac{10 \lambda}{A^2 B} \right) \left(\psi - \frac{U^2}{4 h^2} \right) \right] \end{aligned} \quad (3.122)$$

where the specific energy is

$$e = \frac{U^2}{2} + \frac{\beta h^2 \psi}{2} + K \frac{g h}{2} \cos \theta \quad (3.123)$$

and

$$\Pi = \beta h^3 \psi + K \frac{g h^2}{2} \cos \theta. \quad (3.124)$$

The discontinuities that propagate at speed λ_0 are called contact discontinuities, while the two other are shocks (discontinuities). Across a shock propagating at a celerity a , the quantities $m = h(U - a)$ (conservation of mass) and ψ (conservation of enstrophy) are continuous. Furthermore, the jump of height h is constrained by the shock relation

$$m^2 \left[\frac{1}{h} \right] + \beta \psi [h^3] + \frac{K g \cos \theta}{2} [h^2] = 0, \quad (3.125)$$

where the notation $[f]$ stands for the jump of the quantity f .

It is important to note that the third equation of the system is the enstrophy equation and not the energy equation. This implies that, through a discontinuity, the enstrophy is conserved and the energy is dissipated. In the framework of hyperbolic systems of conservation laws, the energy is the mathematical entropy of the system.

5.3. Linear stability

We study here the linear stability of a steady uniform solution of (3.116). In other words, we suppose that the solutions are sinusoidal perturbations of the constant solution given by $\tilde{h} = 1$, $\tilde{U} = \tilde{U}^{(0)}$ (see (3.74)) and $\tilde{\psi} = \tilde{\psi}^{(0)}$. The depth-averaged velocity can be scaled with its value in the constant solution, which yields $\tilde{A} = 5/2$, $\tilde{U}^{(0)} = 1$ and $\tilde{\psi}^{(0)} = 1/4$. We thus write $\tilde{h} = 1 + h'$, $\tilde{U} = 1 + U'$, $\tilde{\psi} = 1/4 + \psi'$, where $(h', U', \psi') = (a_1, a_2, a_3) \exp[i k(\tilde{x} - c \tilde{t})]$. The amplitudes a_1, a_2, a_3 are three constants, as well as the wave number k and the phase velocity c . We obtain the following linearised system:

$$M(\varepsilon) (h', U', \psi')^T = 0, \quad \text{with} \quad M(\varepsilon) = \frac{\cos \theta}{F^2} R + i \varepsilon k L \quad (3.126)$$

3. Modèle consistant d'écoulement granulaire incompressible – 5. Properties and extensions of the model

where the matrix L comes from the conservative terms of the left-hand side of the equations, and is given by

$$L = \begin{pmatrix} 1-c & 1 & 0 \\ \frac{3\beta}{4} + \frac{K\cos\theta}{F^2} & 1-c & \beta \\ 0 & 0 & 1-c \end{pmatrix}, \quad (3.127)$$

and the matrix R comes from the right-hand side relaxation terms and is given by

$$R = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{3}{2B} - \frac{4\lambda}{5B} & \frac{1}{B} + \frac{4\lambda}{5B} & -\frac{8\lambda}{5B} \\ 3\tilde{\alpha}_1 - \tilde{\alpha}_2 & -2\tilde{\alpha}_1 + \tilde{\alpha}_2 & -2\tilde{\alpha}_2 \end{pmatrix}. \quad (3.128)$$

The dispersion relation is then obtained by imposing the condition $\det(M(\varepsilon)) = 0$. In order to obtain an analytical stability threshold in the limit $\varepsilon k \ll 1$, we expand $c = c^{(0)} + \varepsilon c^{(1)} + O(\varepsilon^2)$, which leads to $L = L^{(0)} + \varepsilon L^{(1)} + O(\varepsilon^2)$. We then obtain a similar expansion for the determinant of $M(\varepsilon)$, that we will denote $\det(M(\varepsilon)) = \delta^{(0)} + \varepsilon \delta^{(1)} + \varepsilon^2 \delta^{(2)} + O(\varepsilon^3)$. We compute for the order 0 that $\delta^{(0)} = \det(\cos\theta R / F^2) = 0$. For the first order in ε , we find, after calculations,

$$\delta^{(1)} = i k \frac{2\cos^2\theta}{BF^4} \left(c^{(0)} - \frac{5}{2} \right) \left(\tilde{\alpha}_2 + \frac{8\lambda}{5} \tilde{\alpha}_1 \right). \quad (3.129)$$

Hence the condition $\delta^{(1)} = 0$ yields $c^{(0)} = 5/2$. The matrix $L^{(0)}$ can thus be rewritten

$$L^{(0)} = \begin{pmatrix} -\frac{3}{2} & 1 & 0 \\ \frac{3\beta}{4} + \frac{K\cos\theta}{F^2} & -\frac{3}{2} & \beta \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}. \quad (3.130)$$

For the order 2 in ε , we obtain

$$\delta^{(2)} = \left[\frac{2ik\cos^2\theta}{BF^4} c^{(1)} - \frac{k^2\cos\theta}{2F^2} \left(-\frac{25}{4} + \frac{(4-3C)\cos\theta}{F^2} \right) \right] \left(\tilde{\alpha}_2 + \frac{8\lambda\tilde{\alpha}_1}{5} \right). \quad (3.131)$$

Solving $\delta^{(2)} = 0$ then yields

$$c^{(1)} = i \frac{kB}{4\cos\theta} \left(\frac{25}{4} F^2 - (4-3C)\cos\theta \right). \quad (3.132)$$

The condition of stability is $\text{Im}(c^{(1)}) < 0$, which gives $25F^2/4 < (4-3C)\cos\theta$. A consistent expression of the stability criterion for the $\mu(I)$ rheology was obtained by Forterre 2006, using a slightly different Froude number $\mathcal{F}$, defined as $\mathcal{F}^2 = F^2 / \cos\theta$. With this definition of the Froude number, the obtained stability criterion is exactly the same as

found by Forterre 2006 and it reads

$$\mathcal{F} < \mathcal{F}_c = \frac{4}{5} \sqrt{1 - \frac{3C}{4}} = \frac{4}{5} \sqrt{1 - \frac{3}{2} \tan^2 \theta + \frac{3}{4} \frac{(\mu_2 - \tan \theta)(\tan \theta - \mu_1)}{\mu_2 - \mu_1} \tan \theta}. \quad (3.133)$$

The stability criteria are identical because the present depth-averaged model is consistent up to order 1 in ε with the $\mu(I)$ rheology.

Note that the critical Froude number $\mathcal{F}_c$ is real if the condition $C < 4/3$ is satisfied. This condition is also a condition for a well-posed enstrophy equation (see §4.3 above) and to guarantee the hyperbolicity of the system (see §5.2). Since this condition appears in the stability criterion of a steady uniform flow for the $\mu(I)$ rheology, it is not specific to the model derived in this work. This condition is discussed in §6.

The variation of the consistent critical Froude number (present model and Forterre 2006) as a function of the inclination is presented in Figure 3.2 (red curve) in the case of glass beads, using the values $\mu_1 = \tan 20.9^\circ$ and $\mu_2 = \tan 32.76^\circ$ (Jop et al. 2005), together with the experimental measurements of Forterre and Pouliquen 2003 (black dots). The stability criterion obtained with the two-equation models with or without diffusion (e.g. Forterre and Pouliquen 2003; Gray and Edwards 2014) is also shown, with a shape factor of 1 (velocity constant over the depth) (black line), or with a shape factor of 5/4 (Bagnold velocity profile) (blue line). These criteria differ from the criterion obtained with the $\mu(I)$ rheology because these Saint-Venant models, both with a shape factor of 1 or 5/4, are inconsistent approximations of the governing equations. The criterion obtained with a constant velocity profile ($\mathcal{F}_c = 2/3$) is paradoxically closer to the consistent criterion than with a shape factor of 5/4 ($\mathcal{F}_c = 2/\sqrt{5} \simeq 0.89$) because some first-order terms were neglected in the derivation of the model. The systematic asymptotic expansions used to derive the present model enables to obtain both a consistent stability criterion and a Bagnold velocity profile for steady uniform flows with its correction for flows in a non-equilibrium state (see also the discussion in §5.5).

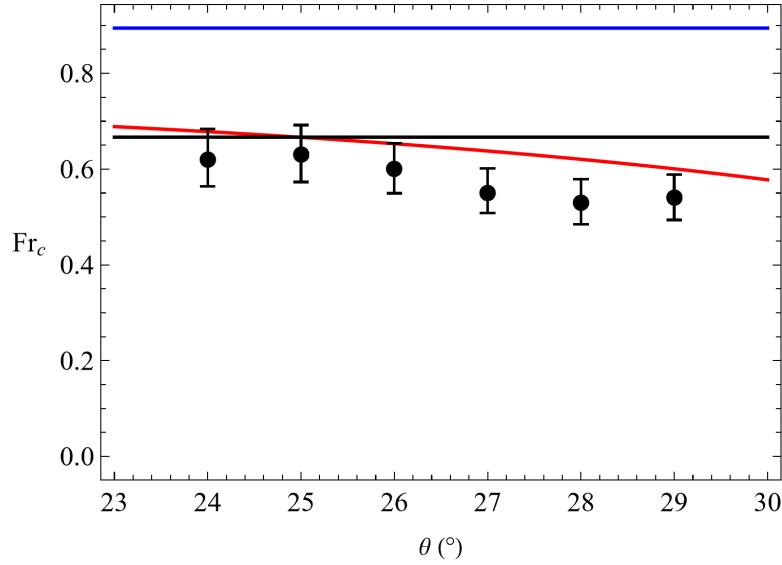


Figure 3.2. – Critical Froude number as a function of the angle θ for glass beads. Theoretical value obtained with the $\mu(I)$ rheology Forterre 2006 and given by the present model (red curve). Saint-Venant model with a shape factor of 1 (black line) and of 5/4 (blue line) (Forterre and Pouliquen 2003). Black dots: experimental measurements of Forterre and Pouliquen 2003.

Using the dispersion relation, it is also possible to express $k = k(\omega, F)$, where $\omega = ck$. It is then possible to study the neutral stability curve defined by $\text{Im}(k) = 0$ (if ω is real). Computations (see Appendix 8.2) show that $\partial F_c / \partial \omega|_{\omega=0} = 0$ and that $\partial^2 F_c / \partial \omega^2|_{\omega=0} = f(\lambda)$, where λ is the parameter introduced in Section 4.2 and $f(\lambda)$ is a complicated function of λ which vanishes when λ has the value

$$\lambda = \frac{847(4 - 3C)}{20(157C + 84)}. \quad (3.134)$$

If we use this expression of λ , the expression (3.133) of the critical Froude number $\mathcal{F}_c$ and the expression $k = 2\omega/5$ of the wavenumber, the dispersion relation is automatically satisfied. This implies that, for this value of λ , the neutral stability curve $F = F(\omega)$ is obtained at the fixed value $\mathcal{F}_c$ of the Froude number independently of the frequency ω and that the phase velocity is also independent of the frequency on the neutral stability curve and equal to 5/2.

The independence of the critical Froude number from the frequency is obtained with all first-order two-equation hyperbolic models in every rheology if only weight, friction and fluxes are included (i.e. with no capillarity, diffusion or any other effect giving higher derivatives). The phase velocity is also independent of the frequency on the neutral stability curve. The proof is given in Appendix 8.3. The behaviour of two-equation models is thus straightforward: the variation of the critical Froude number with frequency is only possible if diffusion is added to the model or other effects modelled with derivatives of high order such as capillarity (see Ruyer-Quil et al. 2002 for the effect of capillarity in a first-order model in the case of Newtonian fluids).

The mathematical structure of three-equation models allows a greater variety of behaviours, even for a first-order model without high-order derivatives. Consistency therefore does not imply model uniqueness, and modelling choices must be made which will determine the behaviour of the model outside the validity domain of the derivation assumptions. The choice is to reproduce with the three-equation model of first order the independence of the critical Froude number with frequency obtained with first-order two-equation models. The chosen expression of λ implies also that the phase velocity is independent of the frequency on the neutral stability curve.

Another interesting feature of the choice (3.134) for λ is that the stability condition can be found by the same argument as for two-equation models. It was shown by Whitham 2011 that the stability condition for these models amounts to a condition on the kinematic wave velocity and the characteristic velocities: the steady uniform flow is stable if the kinematic wave velocity a_0 (whose value is 5/2 in dimensionless form) is between the two characteristic velocities i.e. $\lambda_- < a_0 < \lambda_+$. This condition was applied to the case of granular flow models by Forterre and Pouliquen 2003 and Börzsönyi, Halsey, et al. 2005 in particular. For the three-equation model, the stability condition cannot be found with this simple argument except if λ has the expression (3.134). In this particular case, $\lambda_- < a_0 < \lambda_+$, which in practice reduces to $a_0 < \lambda_+$, gives the condition (3.133).

Moreover, the choice (3.134) is supported by the satisfactory agreement of the spatial growth rate and phase velocity with the theoretical and experimental results and of the high-frequency extrapolation of the critical Froude number (see §5.4 and Figures 3.3, 3.4 and 3.5 below). For these reasons, the particular value (3.134) of λ is the final choice for the model.

5.4. Adding diffusive effects

First-order models such as the present model (3.116) lead to some discrepancies due to the absence of diffusive terms (Forterre 2006; Andreotti et al. 2013; Gray and Edwards 2014). The spatial growth rate in particular is not accurate far from the long-wave limit and the stabilisation of high frequencies is not predicted. Therefore it is important to add diffusive terms to the model. These effects are of second order with respect to ε but, since the second-order full consistency is very complicated and, moreover, does not yield well-posed models, only the linear terms due to the stress gradient in the stream-wise direction are considered. The integration of (3.31) can be written

$$\frac{\partial \tilde{h} \tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} (\tilde{h} \tilde{U}^2 + \tilde{h}^3 \tilde{\psi}) = \frac{1}{\varepsilon \phi F^2} \left(\phi \sin \theta \tilde{h} - \varepsilon \frac{\partial}{\partial \tilde{x}} \int_0^{\tilde{h}} \tilde{p} d\tilde{z} - \tilde{\tau}_{xz}(0) + \varepsilon^2 \int_0^{\tilde{h}} \frac{\partial \tilde{\tau}_{xx}}{\partial \tilde{x}} d\tilde{z} \right). \quad (3.135)$$

In this equation, only the second-order terms given by the integral of $\partial \tilde{\tau}_{xx} / \partial \tilde{x}$ are kept.

Equation (3.52) yields

$$\frac{\partial \tilde{\tau}_{xx}^{(0)}}{\partial \tilde{x}} = \phi \sin \theta \left\{ \left(\frac{\partial \tilde{h}}{\partial \tilde{x}} \right)^2 \left[\left(1 - \frac{\tilde{z}}{\tilde{h}} \right)^{1/2} + \left(1 - \frac{\tilde{z}}{\tilde{h}} \right)^{-1/2} - 2 \right] + 2 \frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} [\tilde{h}^{1/2} (\tilde{h} - \tilde{z})^{1/2} - \tilde{h} + \tilde{z}] \right\}, \quad (3.136)$$

which leads to

$$\int_0^{\tilde{h}} \frac{\partial \tilde{\tau}_{xx}^{(0)}}{\partial \tilde{x}} d\tilde{z} = \frac{1}{3} \phi \sin \theta \frac{\partial}{\partial \tilde{x}} \left(\tilde{h}^2 \frac{\partial \tilde{h}}{\partial \tilde{x}} \right). \quad (3.137)$$

Using the expression (3.74) of $\tilde{U}^{(0)}$, this term can be written as a diffusive term

$$\int_0^{\tilde{h}} \frac{\partial \tilde{\tau}_{xx}^{(0)}}{\partial \tilde{x}} d\tilde{z} = \phi F^2 \frac{\partial}{\partial \tilde{x}} \left(\tilde{v}_{\text{eff}} \tilde{h} \frac{\partial \tilde{U}^{(0)}}{\partial \tilde{x}} \right), \quad (3.138)$$

with an effective viscosity

$$\tilde{v}_{\text{eff}} = \frac{5 \sin \theta}{9 \tilde{A} F^2} \tilde{h}^{1/2}. \quad (3.139)$$

Defining $\tilde{\alpha}_v$ by $\tilde{v}_{\text{eff}} = \tilde{\alpha}_v \tilde{h}^{1/2}$, we can write the depth-averaged momentum balance equation

$$\begin{aligned} \frac{\partial \tilde{h} \tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\tilde{h} \tilde{U}^2 + \beta \tilde{h}^3 \tilde{\psi} + K \frac{\cos \theta \tilde{h}^2}{2 F^2} \right) &= \frac{\cos \theta}{\varepsilon F^2} \left[\tilde{h} (\tan \theta - \mu_b) + \frac{10 \lambda}{\tilde{A}^2 B} \left(\tilde{\psi} - \frac{\tilde{U}^2}{4 \tilde{h}^2} \right) \right] \\ &+ \varepsilon \frac{\partial}{\partial \tilde{x}} \left(\tilde{\alpha}_v \tilde{h}^{3/2} \frac{\partial \tilde{U}}{\partial \tilde{x}} \right). \end{aligned} \quad (3.140)$$

The dimensionless effective viscosity obtained in Gray and Edwards 2014 is similar, apart from a coefficient $1/F^2$ which comes from a different scaling. The expression of the effective viscosity in Forterre 2006 is different but it is consistent at order 0 with (3.139).

We now need to take the diffusion into account in the energy equation. Technical details are given in Appendix 8.4. Since we consider only linear terms, there is some freedom of choice for non-linear terms that inevitably appear either in the enstrophy equation or in the energy equation or both. As for the first-order hyperbolic model, we impose conservation of enstrophy and dissipation of energy. Therefore, the enstrophy equation is written without non-linear terms, which implies dissipative non-linear terms in the energy equation. The enstrophy balance equation is thus written

$$\frac{\partial \tilde{h} \tilde{\psi}}{\partial \tilde{t}} + \frac{\partial \tilde{h} \tilde{U} \tilde{\psi}}{\partial \tilde{x}} = \frac{2 \tilde{U} \cos \theta}{\varepsilon \tilde{h}^2 F^2} \left[\frac{\tilde{\alpha}_1}{\tilde{h}^{1/2}} \left(\tilde{U} - \frac{2}{5} \tilde{A} \tilde{h}^{3/2} \right) + \tilde{\alpha}_2 \left(\tilde{\psi} - \frac{\tilde{U}^2}{4 \tilde{h}^2} \right) \right] + \varepsilon \frac{\partial}{\partial \tilde{x}} \left(\frac{16 \tilde{\alpha}_v}{7 \beta} \tilde{h}^{3/2} \frac{\partial \tilde{\psi}}{\partial \tilde{x}} \right). \quad (3.141)$$

The new depth-averaged energy balance equation is now found from the three equa-

tions of the system and can be written

$$\begin{aligned} & \frac{\partial}{\partial \tilde{t}} \left(\tilde{h} \frac{\tilde{U}^2}{2} + \frac{\beta \tilde{h}^3 \tilde{\psi}}{2} + K \frac{\tilde{h}^2 \cos \theta}{2F^2} \right) + \frac{\partial}{\partial \tilde{x}} \left(\frac{\tilde{h} \tilde{U}^3}{2} + \frac{3\beta \tilde{h}^3 \tilde{U} \tilde{\psi}}{2} + K \frac{\tilde{h}^2 \tilde{U} \cos \theta}{F^2} \right) \\ &= \frac{\tilde{U} \cos \theta}{\varepsilon F^2} \left[\tilde{h} (\tan \theta - \mu_b) + \frac{\beta \tilde{\alpha}_1}{\tilde{h}^{1/2}} \left(\tilde{U} - \frac{2}{5} \tilde{A} \tilde{h}^{3/2} \right) + \left(\beta \tilde{\alpha}_2 + \frac{10\lambda}{\tilde{A}^2 B} \right) \left(\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{h}^2} \right) \right] \\ &+ \varepsilon \frac{\partial}{\partial \tilde{x}} \left(\tilde{\alpha}_v \tilde{h}^{3/2} \tilde{U} \frac{\partial \tilde{U}}{\partial \tilde{x}} \right) + \varepsilon \frac{\partial}{\partial \tilde{x}} \left(\frac{8\tilde{\alpha}_v}{7\beta} \tilde{h}^{7/2} \frac{\partial \tilde{\psi}}{\partial \tilde{x}} \right) - \varepsilon \tilde{\alpha}_v \tilde{h}^{3/2} \left(\frac{\partial \tilde{U}}{\partial \tilde{x}} \right)^2 - \varepsilon \frac{16\tilde{\alpha}_v}{7\beta} \tilde{h}^{5/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} \frac{\partial \tilde{\psi}}{\partial \tilde{x}}. \end{aligned} \quad (3.142)$$

Reverting to the dimensional quantities, the system can be written

$$\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0, \quad (3.143)$$

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} (hU^2 + \Pi) = g \cos \theta \left[h (\tan \theta - \mu_b) + \frac{10\lambda}{A^2 B} \left(\psi - \frac{U^2}{4h^2} \right) \right] + \frac{\partial}{\partial x} \left(\alpha_v h^{3/2} \frac{\partial U}{\partial x} \right), \quad (3.144)$$

$$\frac{\partial h\psi}{\partial t} + \frac{\partial hU\psi}{\partial x} = \frac{2gU \cos \theta}{h^2} \left[\frac{\alpha_1}{h^{1/2}} \left(U - \frac{2}{5} Ah^{3/2} \right) + \alpha_2 \left(\psi - \frac{U^2}{4h^2} \right) \right] + \frac{\partial}{\partial x} \left(\frac{16\alpha_v}{7\beta} h^{3/2} \frac{\partial \psi}{\partial x} \right), \quad (3.145)$$

with the expressions of A and B given in (3.117), of α_1 and α_2 in (3.118), of λ and K in (3.119), of the basal friction μ_b in (3.120), with

$$\alpha_v = \frac{5g \sin \theta}{9A} \quad (3.146)$$

and with

$$\Pi = \beta h^3 \psi + K \frac{g h^2}{2} \cos \theta. \quad (3.147)$$

The system admits the energy balance equation

$$\begin{aligned} & \frac{\partial h e}{\partial t} + \frac{\partial}{\partial x} (hUe + \Pi U) \\ &= gU \cos \theta \left[h (\tan \theta - \mu_b) + \frac{\beta \alpha_1}{h^{1/2}} \left(U - \frac{2}{5} Ah^{3/2} \right) + \left(\beta \alpha_2 + \frac{10\lambda}{A^2 B} \right) \left(\psi - \frac{U^2}{4h^2} \right) \right] \\ &+ \frac{\partial}{\partial x} \left(\alpha_v h^{3/2} U \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial x} \left(\frac{8\alpha_v}{7\beta} h^{7/2} \frac{\partial \psi}{\partial x} \right) - \alpha_v h^{3/2} \left(\frac{\partial U}{\partial x} \right)^2 - \frac{16\alpha_v}{7\beta} h^{5/2} \frac{\partial h}{\partial x} \frac{\partial \psi}{\partial x} \end{aligned} \quad (3.148)$$

with the specific energy e given by (3.123).

The dispersion relation is modified by the diffusive terms. Taking again sinusoidal perturbations of the form $\tilde{h} = 1 + h'$, $\tilde{U} = \tilde{U}^{(0)} + U'$ and $\tilde{\psi} = \tilde{\psi}^{(0)} + \psi'$, the linearized

system now reads

$$M_V(\varepsilon) \begin{pmatrix} h' \\ U' \\ \psi' \end{pmatrix} = 0, \quad \text{with} \quad M_V(\varepsilon) = \frac{\cos \theta}{F^2} R + i\varepsilon k L + \varepsilon^2 k^2 \tilde{\alpha}_v V, \quad (3.149)$$

where R, L are defined in (3.128), (3.127) and V is the matrix of the linearized viscous terms, given by

$$V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{16}{7\beta} \end{pmatrix}. \quad (3.150)$$

The new dispersion relation is $\det(M_V) = 0$. We now suppose that $\omega = ck \in \mathbb{R}$ and that k is complex. The spatial growth rate is then given by $\sigma = -\text{Im}(k)$. The system is stable if and only if $\sigma < 0$.

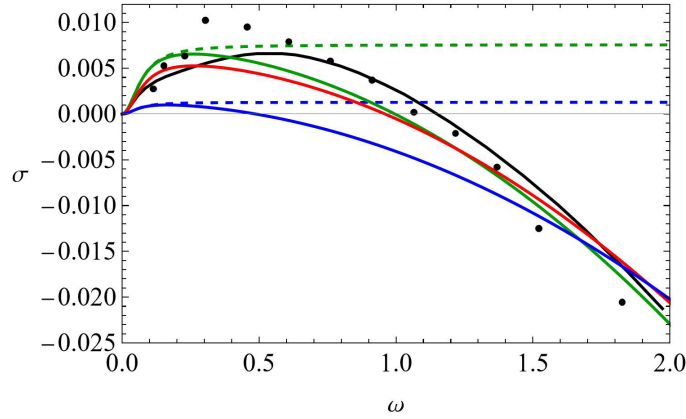


Figure 3.3. – Spatial dimensionless growth rate for $\theta = 29^\circ$ and $\mathcal{F} = 1.02$ as a function of the dimensionless frequency. Black dots: experimental measurements of Forterre and Pouliquen 2003. Red solid curve: present model with diffusion. Black curve : theoretical result for the $\mu(I)$ rheology (Forterre 2006). Two-equation model without diffusion (e.g. Forterre and Pouliquen 2003) with a shape factor $\alpha = 1$ (green dashed curve) and $\alpha = 5/4$ (blue dashed curve) and with diffusion (Gray and Edwards 2014) with a shape factor $\alpha = 1$ (green solid curve) and $\alpha = 5/4$ (blue solid curve).

The variations of the dimensionless spatial growth rate σ with the dimensionless angular frequency is presented in Figure 3.3 for the present model with diffusion (red curve) and λ given by (3.134), in the case of glass beads ($\mu_1 = \tan 20.9^\circ$, $\mu_2 = \tan 32.76^\circ$, Jop et al. 2005) and for $\theta = 29^\circ$ and $\mathcal{F} = 1.02$. The results are compared with the theoretical results obtained for the $\mu(I)$ rheology by Forterre 2006 (black curve) and with the experimental measurements of Forterre 2006 (black dots). The two-equation models without diffusion derived with a constant velocity profile (green dashed curve) and with a Bagnold profile (blue dashed curve) are also presented together with the

model with diffusion (Gray and Edwards 2014) (green curve if $\alpha = 1$, blue curve if $\alpha = 5/4$).

The curve obtained with the present model with diffusion is very close to the curve obtained by (Gray and Edwards 2014). Both models predict quite correctly the stabilization of high frequencies, with a cutoff frequency, and are in reasonable agreement with the experimental measurements and with the theoretical result for the $\mu(I)$ rheology. By contrast, the Saint-Venant models cannot predict the cutoff frequency due to the absence of diffusive terms. These Saint-Venant models are also inconsistent approximations of the governing equations but the inaccuracy is much more important for the model with a shape factor of $5/4$ (Bagnold profile). The present model is also based on a Bagnold velocity profile at leading order but it gives accurate results because of the consistent asymptotic expansions used in its derivation.

The dimensionless phase velocity c is presented in Figure 3.4 (same conditions as for Figure 3.3). The present model with diffusion and λ given by (3.134) (red curve) is in good agreement with the theoretical result for the $\mu(I)$ rheology (black curve) and with the experimental measures (Forterre 2006). The model of Gray and Edwards 2014 with diffusion (green solid curve) gives practically the same result as the model without diffusion (Forterre and Pouliquen 2003) (green dashed curve), and agrees with the theoretical and experimental results if a flat velocity profile is considered, i.e. $\alpha = 1$, while the models with a Bagnold profile ($\alpha = 5/4$), with (blue solid curve) ou without (blue dashed curve) diffusion, give values clearly too large. In this case again, the consistent derivation used for the present model improves the accuracy of the predictions when a Bagnold velocity profile is considered for steady uniform flows. The value of the dimensionless phase velocity for $\omega = 0$ is the celerity $\tilde{A}\tilde{h}^{3/2} = 5/2$ of the kinematic waves while the phase velocity for $\omega \rightarrow \infty$ is the characteristic velocity λ_+ , whose value is $1 + \sqrt{K/F^2 + 3\beta/4} = 2.07$ for the present model if λ is given by (3.134), and $1 + 1/F = 1.98$ for the two-equation models with $\alpha = 1$. The difference between these two high-frequency limit values is due to the Bagnold profile taken into account by the enstrophy, whose equilibrium value is $1/4$ if the equilibrium velocity is 1, and to the consistency factors K and β , which depend on the choice of λ . The critical expression (3.134) of λ gives results in good agreement with the experimental and theoretical results both for the spatial growth rate and the phase velocity.

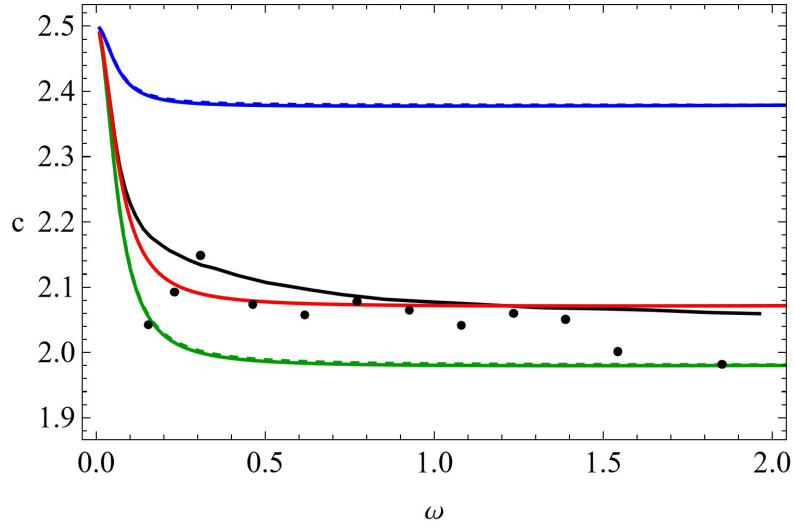


Figure 3.4. – Dimensionless phase velocity for $\theta = 29^\circ$ and $\mathcal{F} = 1.02$ as a function of the dimensionless frequency. Black dots: experimental measurements of Forterre and Pouliquen 2003. Red solid curve: present model with diffusion. Black curve : theoretical result for the $\mu(I)$ rheology (Forterre 2006). Two-equation model without diffusion (e.g. Forterre and Pouliquen 2003) with a shape factor $\alpha = 1$ (green dashed curve) and $\alpha = 5/4$ (blue dashed curve) and with diffusion (Gray and Edwards 2014) with a shape factor $\alpha = 1$ (green solid curve) and $\alpha = 5/4$ (blue solid curve).

The variations of the dimensionless cutoff angular frequency ω_c as a function of the Froude number above criticality $\mathcal{F} - \mathcal{F}_c$ is presented in Figure 3.5 for glass beads and $\theta = 29^\circ$. As pointed out above, there is no uniqueness of a first-order consistent model. The value of the parameter λ is a modelling choice. The choice that has been made, detailed above at the end of §5.3, is to have a critical Froude number independent of frequency for the model without diffusion. This choice leads to the critical value (3.134).

Other special values of λ include $\lambda = 0$ and $\lambda = 11/4$. In the case $\lambda = 0$, there is no relaxation term for the enstrophy in the momentum balance equation and the first-order correction to the granular friction law is entirely written as terms in the momentum flux. In the case $\lambda = 11/4$, we have $\beta = 1$, which means that there is no correction on the shear term $h^3\psi$ in the momentum flux although, since $K < 1$, there is still a correction on the pressure term of the momentum flux (the value of λ needed to have $K = 1$ is $21/4$, so even higher). In the final choice (3.134), part of the first-order correction is written in the momentum flux, with a correction on both the enstrophy and pressure terms, and another part as a relaxation term.

The effect of the parameter λ is shown with the model with diffusion in these three cases: λ is equal to the critical value (3.134) (red solid curve), $\lambda = 0$ (blue curve) and $\lambda = 11/4$ (green curve). Since all these values of λ give a consistent model, they lead to the same result in the validity domain of the shallow-water approximation, thus for small frequencies, but this domain is quite small ($\bar{\omega} < 0.1$ approximately). For larger

frequencies, the cutoff frequency is sensitive to the choice of λ . For the critical value of λ (3.134), the agreement of the model with diffusion with the theoretical result for the $\mu(I)$ rheology (black curve) and with the experimental results (black dots), both obtained by Forterre 2006, is quite good. For smaller values of λ , and in particular for $\lambda = 0$, there are unstable frequencies that are smaller than the cutoff at the long-wave limit, which does not seem to be observed experimentally. For larger values of λ , the cutoff frequency is too small. The diffusion is again important to stabilize the high frequencies. Only the case $\theta = 29^\circ$ is presented, but other values of θ studied by Gray and Edwards 2014 and Forterre 2006 show the same trends.

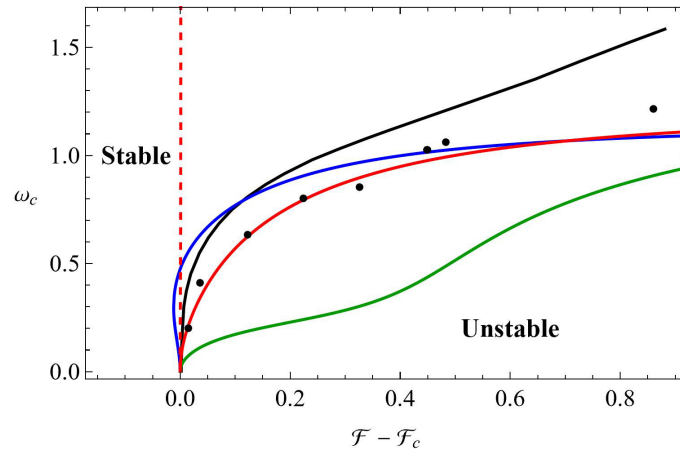


Figure 3.5. – Dimensionless cutoff frequency ω_c of the instability as a function of the Froude number above criticality $\mathcal{F} - \mathcal{F}_c$ for $\theta = 29^\circ$. Black dots: experimental measurements of Forterre and Pouliquen 2003. Present model with diffusion for λ given by (3.134) (red solid curve), $\lambda = 0$ (blue curve) and $\lambda = 11/4$ (green curve). Red dashed curve: present model without diffusion for λ given by (3.134). Black curve: theoretical result for the $\mu(I)$ rheology (Forterre 2006).

5.5. Comparisons with the model of Gray and Edwards 2014

The model of Gray and Edwards 2014 with diffusion was used for many applications (see §1). It is therefore important to compare the model derived in this article with this model, highlighting the differences and similarities between these two models.

The model of Gray and Edwards 2014 was proposed with a shape factor α in the momentum flux, defined by $\langle u^2 \rangle = \alpha \langle u \rangle^2$. Two values of α were considered: The value $\alpha = 5/4$ corresponds to a Bagnold velocity profile while the value $\alpha = 1$ corresponds to a constant velocity in the depth. Although this difference in the value of alpha seems minor, it induces a significant change in mathematical structure and behaviour, to the extent that these two values correspond to two different models. In most applications, $\alpha = 1$ was the only value used.

Whatever the value of α , the derivation of the model of Gray and Edwards 2014 is inconsistent, in the sense that first-order corrections have been neglected. In

particular, gravity balances friction at leading order, and their difference is of $O(\varepsilon)$. To achieve the first-order accuracy required for consistent modelling of variable flows, this difference cannot be neglected and the corresponding first-order correction must be included in the model. As a result, the instability threshold for steady uniform flows calculated with the model of Gray and Edwards 2014, as with the model of Pouliquen and Forterre 2002, differs from the threshold predicted by the $\mu(I)$ rheology calculated at the long-wave limit by Forterre 2006. By contrast, the present model is first-order consistent and predicts exactly the same instability threshold as the $\mu(I)$ rheology in the long-wave limit.

The discrepancy of the instability threshold predicted by the model of Gray and Edwards 2014, or any other similar two-equation model, can be particularly large if $\alpha = 5/4$. It is more reasonable if $\alpha = 1$. In the case of glass beads down a slope of 29° , the measurements of Forterre and Pouliquen 2003 gave a value of the critical Froude number $\mathcal{F}_c = 0.54 \pm 0.02$ at the zero-frequency limit, whereas the $\mu(I)$ rheology and the present model predict $\mathcal{F}_c = 0.60$, giving a relative deviation of 11%. The models of Gray and Edwards 2014 and Pouliquen and Forterre 2002 give $\mathcal{F}_c = 2/3$ with $\alpha = 1$ (relative deviation: 23%) and $\mathcal{F}_c \simeq 0.89$ if $\alpha = 5/4$ (relative deviation: 65%).

The consistency of the derivation of the present model enables us to reconstruct the velocity profile in the depth. This profile is a Bagnold profile for a steady uniform flow, but the reconstruction allows us to calculate the deviations from this profile due to first-order corrections when the flow is variable. This reconstruction is not possible with the model of Gray and Edwards 2014, particularly if $\alpha = 1$, which corresponds to a constant velocity profile for steady uniform flows. If $\alpha = 5/4$, the equilibrium profile is a Bagnold profile, but the corrections for variable flows cannot be calculated. One of the advantages of introducing the enstrophy variable is that the velocity profile is not fixed as with a shape factor. With a shape factor of $5/4$, the velocity profile is always a Bagnold profile, even for a variable flow. With the present model, thanks to the enstrophy variable, the profile is a Bagnold profile for a steady uniform flow but the model takes into account and calculates the corrections to this profile in the case of a variable flow.

With the model of Gray and Edwards 2014 with $\alpha = 5/4$, no energy balance equation can be obtained (see the general proof in Richard and Gavrilyuk 2012 Appendix A). By contrast, the model with $\alpha = 1$ and the present model are well-posed because both admit an energy balance equation. This is an important feature for physical reasons but also from a numerical point of view, since the energy conservation is essential to prove the stability of numerical schemes. The characteristics of the model with $\alpha = 5/4$ are also completely different from those of the model with $\alpha = 1$. They are also much more complicated. For these reasons, the mathematical structure of the model of Gray and Edwards 2014 changes radically when the value of the shape factor increases from 1 to $5/4$.

The predictions and the mathematical structure of the model of Gray and Edwards 2014 are paradoxically better if a constant velocity profile is assumed (i.e. if $\alpha = 1$) than if a Bagnold profile is taken into account ($\alpha = 5/4$). This paradox, noted by Forterre and Pouliquen 2003, is resolved by the present model, which obtains both accurate results

and a Bagnold profile at equilibrium, thanks to the consistency of its derivation. Since it is hyperbolic (disregarding the diffusive terms), in conservative form, and with an exact energy conservation equation, the present model has also a good mathematical structure.

The diffusion term is the same in the present model and in the model of Gray and Edwards 2014. Since this term is of second order, it can be consistently calculated assuming a Bagnold velocity profile (zero-order solution). In this respect, the present work confirms the expression already found by Gray and Edwards 2014 for the diffusive term. It follows that properties depending on the diffusive term, such as the variations of the spatial growth rate or of the phase velocity with frequency, are equivalent with this model and the model of Gray and Edwards 2014 with $\alpha = 1$. With the value $\alpha = 5/4$, the two-equation models, even with diffusion, lead to a large discrepancy in the spatial growth rate and the phase velocity (see Figures 3.3 and 3.4).

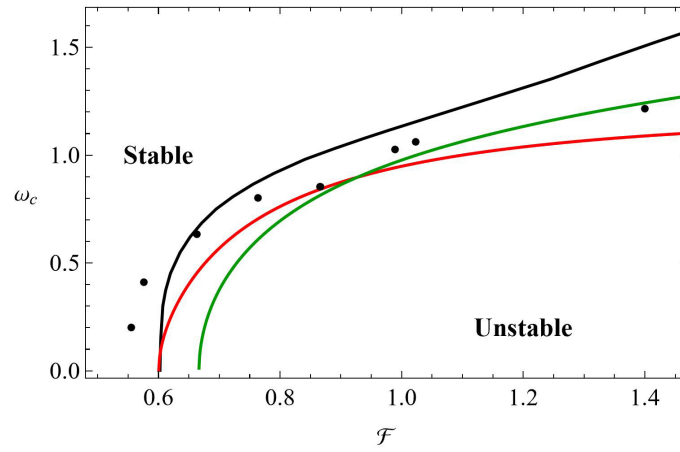


Figure 3.6. – Dimensionless cutoff frequency ω_c of the instability as a function of the Froude number $\mathcal{F}$ for $\theta = 29^\circ$. Black dots: experimental measurements of Forterre and Pouliquen 2003. Present model with diffusion for λ given by (3.134) (red solid curve). Green curve: model of Gray and Edwards 2014 with $\alpha = 1$. Black curve: theoretical result for the $\mu(I)$ rheology (Forterre 2006).

The model of Gray and Edwards 2014 with $\alpha = 1$ and the present model are compared with regard to the variation of the cutoff frequency with the Froude number. In Figure 3.5, all variants of the model were consistent to the $\mu(I)$ rheology. Variations in cutoff frequency could be studied as a function of the difference between the Froude number and its critical value. In this comparison, the critical Froude number predicted by both models are different. The variation of the cutoff frequency as a function of the Froude number $\mathcal{F}$ is shown in Figure 3.6 with the curve predicted by the $\mu(I)$ rheology calculated by Forterre 2006 (black), the experimental measurements of Forterre and Pouliquen 2003 (black dots), the present model (red curve) and the model of Gray and Edwards 2014 (green curve). The main difference between the two models is the value of the critical Froude number i.e. the beginning of the curves

at the limit of a zero frequency. Since the diffusive term is the same in both models, and the variations in cutoff frequency with Froude number are due to the diffusive term, both models predict fairly similar variations apart from the starting point of the curves.

The mathematical structure of the model of Gray and Edwards 2014 with $\alpha = 1$ is the structure of the Saint-Venant equations (or nonlinear shallow-water equations) with an additional diffusive term. The structure of the present model is the structure of the compressible Euler equation with source terms and additional diffusive terms. Both structures are well-known and can be solved numerically with classical and reliable numerical schemes. We do not recommend solving numerically the model with $\alpha = 5/4$, as no guarantee can be given regarding the numerical scheme.

5.6. Numerical resolution

In this subsection, we solve numerically the model in order to study the roll waves instability. The numerical solution is calculated with a Godunov type scheme (HLLC Riemann solver). We solve system (3.116) in the conservative variables $(h, hU, h\psi)$. In these variables, the total energy plays the role of a mathematical entropy for the system. The relaxation source terms are implemented explicitly. The second-order diffusion terms are approximated by centered finite differences. We use the following parameters: $\mu_1 = \tan(20.9^\circ)$, $\mu_2 = \tan(32.76^\circ)$, $I_m = 0.279$ (see Jop et al. 2006), $d = 5 \times 10^{-4}$ m, $\theta = 25^\circ$, $\phi = 0.6 - 0.2 I^{(0)}$ (see (3.45)), $g = 9.81$ m/s, and $h_0 = 10^{-2}$ m. It follows that $\mathcal{F} \approx 0.806 > \mathcal{F}_c \approx 0.666$.

The domain has a length of 100m. The initial conditions are a steady uniform flow of depth h_0 , average velocity $U_0 = 2Ah_0^{3/2}/5$ and enstrophy $\psi_0 = U_0^2/(4h_0^2)$. The boundary condition at the domain entrance is a small sinusoidal perturbation of the depth $h(0, t) = h_0[1 + \alpha_0 \sin(2\pi f t)]$. The relative amplitude and frequency of the perturbation are $\alpha_0 = 0.01$ and $f = 0.316$ Hz respectively. The inlet values of the velocity and enstrophy are $U(0, t) = h_0 U_0 / h(0, t)$ and $\psi(0, t) = (h_0 U_0)^2 / (4h(0, t)^4)$ respectively. The results are presented with a cell size of 0.625mm in order to reach numerical convergence with the diffusive model and a numerical scheme of order 1.

In the beginning, the amplitude of the perturbation increases exponentially. After some time, the amplitude converges to a maximal value. A steady state is then reached and the solution becomes periodic. This is illustrated by Figure 3.7 which shows the height of the flow, for the system (3.116) (purely hyperbolic) and for the system (3.143)–(3.145) (with diffusion terms). The depth profiles of the two system are similar. It can be seen that, in presence of diffusion, the amplitude of the waves is slightly smaller and the propagation is slightly slowed. As expected, the shock is steeper in the purely hyperbolic model. However, the effects of the diffusive terms seem to have little overall impact on the global shape of the solution. This is coherent with the fact that the diffusive terms are of order ε^2 in our asymptotic. Figure 3.8 shows the values taken by the basal friction μ_b and the enstrophy ψ over a period, once the steady state is reached. The basal friction decreases when the depth increases, while the enstrophy increases. If the inlet perturbation is smaller, the length required to reach the steady

state is longer but the steady-state waves are the same.

Finally, the horizontal velocity profile is reconstructed in Figure 3.9, with the help of Equation (3.115). The reconstructions are presented for three different values of x . The black curve is obtained by reconstruction of the velocity at the wave trough. It is very close to a Bagnold profile (solution at equilibrium). The red curve is obtained by reconstruction at the wave crest. The blue curve is obtained at an intermediate state. As the height increases, the profile becomes more and more curved. The first-order correction with respect to the leading-order velocity profile, which represents the deviation of the velocity profile from the Bagnold profile, can be visualized by the difference between a curve (red or blue) and the black curve.

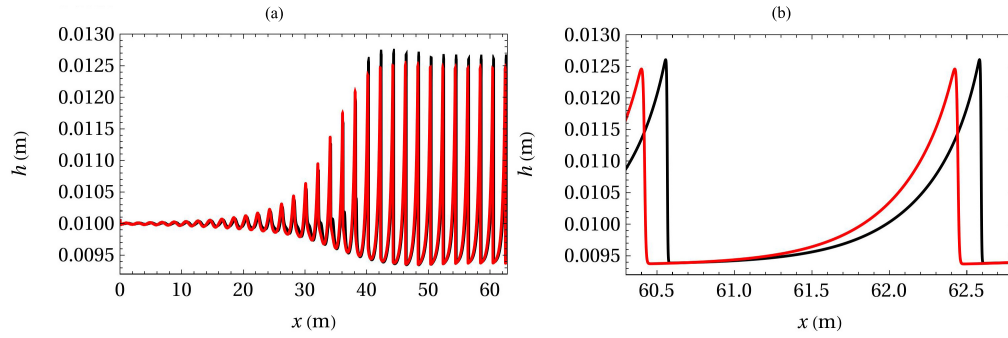


Figure 3.7. – (a) Profile of height for the roll-wave instability. (b) Zoom on the final periodic waves. In black: System (3.116) (purely hyperbolic). In red: System (3.143)–(3.145) (system with diffusion).

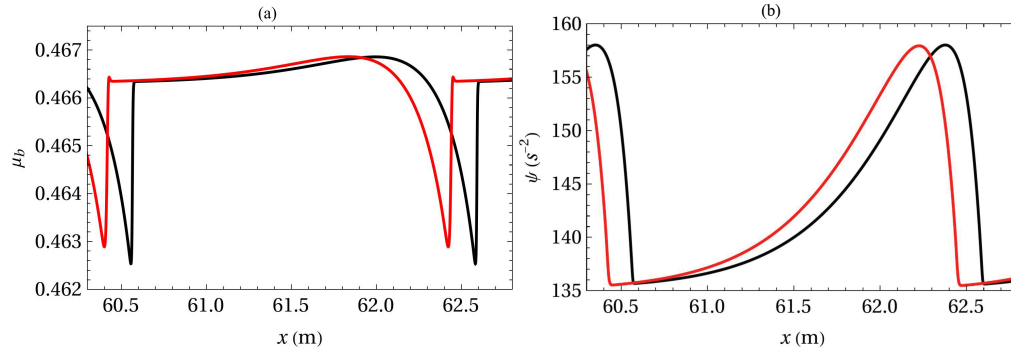


Figure 3.8. – Profile of μ_b (a) and of ψ (b) for the roll-wave instability. In black: System (3.116) (purely hyperbolic). In red: System (3.143)–(3.145) (system with diffusion).

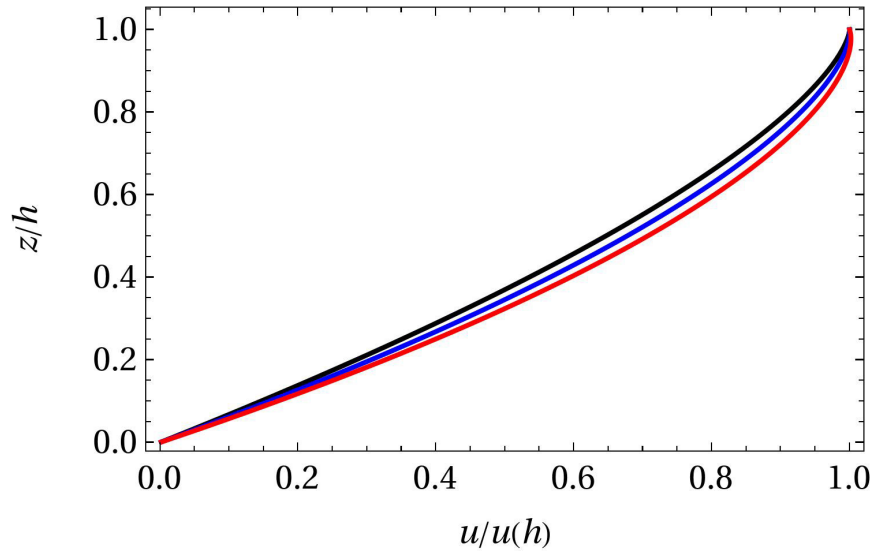


Figure 3.9. – Reconstruction of the velocity profile for the numerical simulation of Section 5.6. In black, profile at the wave trough (just after the shock). In red, profile at the wave crest. In blue, profile where the height of the perturbation reaches eighty percent of the maximal height.

6. Limitations of the model

It is essential to regularize the model for certain values of the angle θ because some quantities of the system (3.143)–(3.145) may change sign or become infinite, resulting in a ill-posed model.

In order to understand why this can happen, let us recall some facts about the $\mu(I)$ rheology. As mentioned in the first section, this rheology corresponds to a friction law, for which the friction coefficient is given by

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + I_m/I}. \quad (3.151)$$

The two parameters μ_1 and μ_2 are determined experimentally. They characterize the angles for which steady uniform flows can be observed. These steady uniform flows only exist when

$$\mu_1 < \tan \theta < \mu_2. \quad (3.152)$$

Outside this range, there is no equilibrium and all the formulas that we obtained previously do not make any sense a priori. We insist on the fact that the model (3.143)–(3.145) is derived with an asymptotic method, under the hypotheses that θ is constant and inside this range, so that the flow remains close to the uniform stationary flow. However, for geophysical applications, the value of the slope may not lie within the interval $[\mu_1, \mu_2]$.

Recall that the expression of the inertial number I at leading order is

$$I^{(0)} = I_m \frac{\tan \theta - \mu_1}{\mu_2 - \tan \theta}. \quad (3.153)$$

This expression is only valid when the condition (3.152) is verified and makes no sense otherwise.

- In the limit $\tan \theta \rightarrow \mu_1$, we have $I^{(0)} \rightarrow 0$. It follows that $U^{(0)}$ and $\psi^{(0)}$ both converge to 0 and the flow should stop. This is called the liquid/solid transition. In the right-hand side of the system (3.143)–(3.145), the relaxation coefficients as well as the diffusion coefficients diverge to $+\infty$, which indicates that U and ψ should go to zero instantaneously. Given that this is likely to cause problems for numerical resolution and that the instantaneous stopping of an avalanche is not physical, these coefficients need to be regularized. In addition, the frictional hysteresis is not taken into account by the model derivation assumptions.
- In the limit $\tan \theta \rightarrow \mu_2$, $I^{(0)} \rightarrow +\infty$. The relaxation and diffusion coefficients go to zero. This is the liquid/gas transition, after which the flow becomes uniformly accelerated and enters a kinetic regime in which the volume fraction decreases. For $\tan \theta > \mu_2$, the effective viscosity becomes negative, requiring regularization. Beyond this limit, the flow is no longer correctly described either by the $\mu(I)$ rheology or, consequently, by the model, which means that the equations have to be modified.
- Finally, another singularity of the model can appear when the coefficient μ_2 is large. Indeed, recall that the stability criterion in the long-wave limit is

$$\mathcal{F}^2 < \mathcal{F}_c^2 = \frac{16}{25}(4 - 3C), \quad (3.154)$$

where $C = C(\theta)$ is an increasing function of θ which is equal to $2\mu_2^2$ when $\tan \theta = \mu_2$. If $\mu_2 < \sqrt{2/3}$, the condition $C < 4/3$ is automatically satisfied for a slope in the range $[\mu_1, \mu_2]$. If $\mu_2 > \sqrt{2/3} \approx 0.816$, there are some values of θ which verify (3.152) for which every steady flow is unstable, even when $\mathcal{F} = 0$, because $C > 4/3$ (which gives $\mathcal{F}_c^2 < 0$!). This phenomenon is inherent to the $\mu(I)$ rheology since the stability criterion is the same for the bulk equations, as shown by Forterre 2006. In the system (3.143)–(3.145), it is materialised by the fact that the sign of the relaxation coefficient α_2 changes when the critical Froude number is equal to zero (see §4.3 above), as well as the sign of the coefficient K in front of the hydrostatic pressure and of the potential energy term in the expression (3.123) of the energy. In particular, the enstrophy ψ grows exponentially in this case and the system is ill-posed. Furthermore, the hyperbolicity of the system is not guaranteed anymore when $C > 4/3$ (see §5.2). It is not clear whether this instability effect has a real physical meaning or if it is only a shortcoming of the rheology. In the literature, the value of μ_2 is often lower than the critical value $\sqrt{2/3}$ ($\mu_2 = \tan 32.76^\circ \approx 0.643$ in the paper of Jop et al. 2006 for instance). However, the value of μ_2 was $\mu_2 = 1$ for the experiments

of Börzsönyi and Ecke 2006. They observed steady flows only under a critical angle $\tan\theta \approx 0.85$, above which the flow becomes uniformly accelerated. With their parameters, this corresponds to a value of $C(\theta) \approx 1.36$, which is close to the theoretical value $C = 4/3 \approx 1.33$ that we obtain for the instability of the model. Further investigations need to be performed to see whether this is a coincidence or a successful prediction of the $\mu(I)$ rheology. In any case, this instability also requires regularization.

We use the term “regularization” here in a very broad sense. By regularization, we mean a shift in the model towards physical behaviours obeying laws and assumptions that can be very different, and not just mathematical procedures. The physical phenomena that appear outside the model’s domain of validity are complex and not always well known. How to capture them in the model is also a complex problem that does not necessarily have a unique solution. A regularized system should be usable for numerical simulations with any value of the slope. However, the development of such a regularized model requires an in-depth study of flow stopping, of accelerated flows and of granular avalanche fronts, which is beyond the scope of this article and will be the subject of a future work.

In particular, the flow near the head of a granular front does not meet the assumptions of a shallow flow. Consequently, the characteristics of such a flow may differ greatly from a steady uniform flow, with the result that the asymptotic method may be irrelevant. Saingier et al. 2016 and Lagrée, Saingier, et al. 2017 showed that, while a Bagnold velocity profile was in good agreement with experimental results, the shape coefficient tends towards $\alpha = 1$ near the head of the front. This is the same value as for a constant velocity in depth. This raises the question of the velocity profile in this part of the flow. The model used in this work is therefore not yet suitable for treating granular avalanche fronts and, for this case too, a regularization will be necessary.

The ill-posed nature of the model for liquid/solid or liquid/gas transitions is a consequence of the limitations of the $\mu(I)$ rheology to these same transitions. The limitations of the rheology for quasi-static regimes have been highlighted by many authors (see e.g. GdR MiDi 2004; Andreotti et al. 2013). For these regimes, the simple assumption of a local rheology is not valid. Large spatial correlation lengths diverging at the quasi-static limit, as well as mechanical noise, could explain the appearance of non-local effects. Various non-local laws have been proposed for the rheology (see e.g. Pouliquen and Forterre 2009; Kamrin and Koval 2012; Kamrin and Henann 2015; Bouzid et al. 2015; Kamrin 2019; Gaume et al. 2020). The dilute regime at the transition to the gaseous regime described by the kinetic theory of granular gases is not predicted by the $\mu(I)$ rheology either. It is therefore not surprising that a model consistent with the $\mu(I)$ rheology is characterised by the same limitations as the rheology itself.

On the other hand, the limitations of the model are not directly related to the ill-posed behaviour of the $\mu(I)$ rheology found by Barker, Schaeffer, Bohórquez, et al. 2015 at the high-wavenumber limit. The model is derived at the long-wave limit and has a mathematical structure similar to the structure of the compressible Euler equations with source terms and diffusion terms. Even the possible singularity which can arise if $\mu_2 > \sqrt{2/3}$ is found at the long-wave limit as in Forterre 2006. The model is always well-

posed for $\mu_1 < \tan \theta < \mu_2$ if $\mu_2 < \sqrt{2/3}$ and for $\mu_1 < \tan \theta < \sqrt{2/3}$ if $\mu_2 > \sqrt{2/3}$, which is a wider range of stability than the range found in Barker, Schaeffer, Bohórquez, et al. 2015. It follows that the depth-averaging process at the long-wave limit eliminates the ill-posed behaviour at high wavenumbers found by Barker, Schaeffer, Bohórquez, et al. 2015, who noted that “it is therefore possible to push the depth-averaged $\mu(I)$ rheology somewhat further than the full rheology” and this has allowed to calculate roll waves “using the depth-averaged theory, without getting into issues of ill-posedness”.

7. Conclusion

In this work, we derive a new depth-averaged model for an incompressible granular flow described by the $\mu(I)$ rheology. The derivation is made using a consistent asymptotic method under the assumption of shallow flows. This method implies that, unlike the models derived to date for the $\mu(I)$ rheology such as Forterre and Pouliquen 2003 or Gray and Edwards 2014, the model is consistent at the first order, with respect to the small shallow-flow parameter, with the $\mu(I)$ rheology. In particular, for the first time, the first-order correction to the classical granular friction law is included in the model. For steady uniform flows, the model is consistent with a Bagnold velocity profile, which is found at leading order. Otherwise the model includes the first-order correction to this profile. This model resolves the paradox of previous models, for which the inclusion of a Bagnold profile led to a much greater disagreement with experimental and theoretical results than a constant velocity in the depth. Since the model is consistent, and in contrast to previous models, it gives the exact stability criterion of the $\mu(I)$ rheology in the long-wave limit as found by Forterre 2006.

The resulting three-equation model is a hyperbolic system of conservation laws for the flow depth, the depth-averaged velocity and an additional quantity, called enstrophy, which takes into account shear effects. This system admits an exact energy balance equation. The model has the classical structure of the compressible Euler equations with relaxation source terms, which is well-suited for numerical resolution. In contrast to the two-equation models with a shape factor of 5/4, this three-equation structure remains well-posed while taking into account a Bagnold velocity profile for steady uniform flows.

Thanks to the consistency of the asymptotic method used for the derivation, the variation in the depth of the velocity can be reconstructed explicitly with the three variables of the depth-averaged model. The deviations to the Bagnold profile can thus be calculated for flows which are not steady and uniform, such as granular roll waves.

This hyperbolic model is also extended to include diffusive effects. We confirm the expression of the effective viscosity obtained by Gray and Edwards 2014, which is indeed the consistent form resulting from the application of the asymptotic method. The spatial growth rate and the phase velocity predicted by this extended model is in good agreement with both the theoretical predictions of the $\mu(I)$ rheology and with the experimental data of Forterre and Pouliquen 2003. In particular, the extended model predicts successfully the stabilization of high frequencies and the variation of

the cutoff frequency with the Froude number.

In this respect, it is important to note that it is possible to derive an infinite number of consistent models, all of which are equivalent within the validity domain of the long-wave approximation, but differ outside this domain. Outside the domain of validity of the shallow-flow assumption, the three-equation structure can produce a greater variety of behaviours than the two-equation structures. The best variant among the different possible consistent models was selected to obtain a critical Froude independent of the frequency for the model without diffusion, as in the case of non-diffusive two-equation models. In this variant, the instability threshold can be found by writing that the kinematic wave velocity lies between the maximum and minimum characteristic velocities, again as for two-equation models. This choice is also supported by the high-frequency extrapolation of the model with diffusion regarding the spatial growth rate, the phase velocity and the critical Froude number.

Numerical simulations of granular roll waves are presented, using classical finite-volume schemes with an approximate Riemann solver. All depth-averaged quantities can be calculated and the velocity profiles in the depth can be easily reconstructed.

The asymptotic method used to derive the model assumes that the flow remains close to the steady uniform flow and therefore that such a steady uniform flow exists. In the context of the $\mu(I)$ rheology, a steady uniform flow is only possible within a certain range of the angle of inclination of the bottom with respect to the horizontal. This range defines the model's domain of validity. Outside this range, certain model quantities become infinite or change sign, and the model becomes mathematically ill-posed. This problem is related to the solid/liquid and liquid/gas transitions, which are well-known limits of the $\mu(I)$ rheology. A regularization of the model and different physical assumptions are therefore necessary for these transitions, based on the study of flow stopping, accelerated flows and avalanche fronts, which are not described by the $\mu(I)$ rheology. It will be the subject of a future work.

The model has been derived under the assumption of incompressible flows, but dilatancy phenomena can be significant in practice. A further development of this work will be to take into account variations in volume fraction with the inertial number according to the phenomenological expressions of the local rheology proposed in the literature. A further development will concern granular flows on smooth bottoms, for which a non-zero slip velocity on the bottom is possible and can play an important role, in particular for granular jumps.

8. Appendix

8.1. Derivation of the depth-averaged energy equation

We compute the right-hand-side of (3.100). In order to do this, we use the expansion (3.90) to keep track of the friction law (3.88). We first compute

$$\begin{aligned}\frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} &= \frac{\partial}{\partial \tilde{z}} (\tilde{\tau}_{xz}^{(0)} + \varepsilon \tilde{\tau}_{xz}^{(1)}) + O(\varepsilon^2) = -\mu^{(0)} \phi \cos \theta + \varepsilon \frac{\partial \tilde{\tau}_{xz}^{(1)}}{\partial \tilde{z}} + O(\varepsilon^2) \\ &= -\mu_b \phi \cos \theta + \varepsilon \phi \cos \theta \frac{\tilde{U}^{(1)}}{B \tilde{U}^{(0)}} + \varepsilon \frac{\partial \tilde{\tau}_{xz}^{(1)}}{\partial \tilde{z}} + O(\varepsilon^2),\end{aligned}\quad (3.155)$$

where the last equality comes from (3.90). We obtain with (3.48) and (3.60)

$$\begin{aligned}\int_0^{\tilde{h}} \tilde{u} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} d\tilde{z} &= \int_0^{\tilde{h}} \tilde{u} \phi \cos \theta \left(-\mu_b + \varepsilon \frac{\tilde{U}^{(1)}}{B \tilde{U}^{(0)}} \right) d\tilde{z} + \varepsilon \int_0^{\tilde{h}} \tilde{u}^{(0)} \frac{\partial \tilde{\tau}_{xz}^{(1)}}{\partial \tilde{z}} d\tilde{z} + O(\varepsilon^2) \\ &= \phi \tilde{h} \cos \theta \left[-\tilde{U} \mu_b + \varepsilon \frac{\tilde{U}^{(1)}}{B} + \varepsilon \tilde{A} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{2}{5} \tilde{h}^{3/2} - \frac{F^2 \tilde{A}^2 \tilde{h}^{7/2}}{10 \cos \theta} \right) \right] + O(\varepsilon^2).\end{aligned}\quad (3.156)$$

We compute

$$\frac{\tilde{U}^{(1)}}{B} + \tilde{A} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{2}{5} \tilde{h}^{3/2} - \frac{F^2 \tilde{A}^2}{10 \cos \theta} \tilde{h}^{7/2} \right) = \tilde{A} \frac{\partial \tilde{h}}{\partial \tilde{x}} \frac{3C}{10} \tilde{h}^{3/2} \quad (3.157)$$

We thus obtain the following equation for the energy:

$$\begin{aligned}\frac{\partial}{\partial \tilde{t}} \left(\tilde{h} \frac{\tilde{U}^2}{2} + \frac{\tilde{h}^3 \tilde{\psi}}{2} + \frac{\tilde{h}^2 \cos \theta}{2F^2} \right) &+ \frac{\partial}{\partial \tilde{x}} \left(\frac{\tilde{h} \tilde{U}^3}{2} + \frac{3}{2} \tilde{h}^3 \tilde{U} \tilde{\psi} + \frac{\tilde{h}^2 \tilde{U}}{F^2} \cos \theta \right) \\ &= \frac{\tilde{h} \tilde{U} \cos \theta}{\varepsilon F^2} (\tan \theta - \mu_b) + \frac{\tilde{A}^3 \tilde{h}^{9/2}}{125} \frac{\partial \tilde{h}}{\partial \tilde{x}} + \frac{3}{10} \tilde{A} C \tilde{h}^{5/2} \frac{\cos \theta}{F^2} \frac{\partial \tilde{h}}{\partial \tilde{x}} + O(\varepsilon).\end{aligned}\quad (3.158)$$

To ensure compatibility between the energy equation and the momentum equation, we need to modify the left-hand side of (3.158). We compute with (3.74), (3.79) and (3.84)

$$\frac{1}{2} \frac{\partial \tilde{h}^3 \tilde{\psi}}{\partial \tilde{t}} + \frac{3}{2} \frac{\partial \tilde{h}^3 \tilde{U} \tilde{\psi}}{\partial \tilde{x}} = \frac{13}{250} \tilde{A}^3 \tilde{h}^{9/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} + O(\varepsilon). \quad (3.159)$$

Similarly,

$$C \left(\frac{3}{4} - \frac{\lambda}{7} \right) \left(\frac{\partial}{\partial \tilde{t}} \frac{\tilde{h}^2 \cos \theta}{2F^2} + \frac{\partial}{\partial \tilde{x}} \frac{\tilde{h}^2 \tilde{U} \cos \theta}{F^2} \right) = \frac{2C}{5} \left(\frac{3}{4} - \frac{\lambda}{7} \right) \tilde{A} \tilde{h}^{5/2} \frac{\cos \theta}{F^2} \frac{\partial \tilde{h}}{\partial \tilde{x}} + O(\varepsilon). \quad (3.160)$$

3. Modèle consistant d'écoulement granulaire incompressible – 8. Appendix

We now subtract Equations $(1 - \beta)(3.159) + (3.160)$ from (3.158) to obtain

$$\begin{aligned} & \frac{\partial}{\partial \tilde{t}} \left(\tilde{h} \frac{\tilde{U}^2}{2} + \frac{\beta \tilde{h}^3 \tilde{\psi}}{2} + K \frac{\tilde{h}^2 \cos \theta}{2F^2} \right) + \frac{\partial}{\partial \tilde{x}} \left(\frac{\tilde{h} \tilde{U}^3}{2} + \frac{3\beta \tilde{h}^3 \tilde{U} \tilde{\psi}}{2} + K \frac{\tilde{h}^2 \tilde{U} \cos \theta}{F^2} \right) \\ &= \frac{\tilde{h} \tilde{U} \cos \theta}{\varepsilon F^2} (\tan \theta - \mu_b) + \frac{13\beta - 11}{250} \tilde{A}^3 \tilde{h}^{9/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} + \frac{2\lambda}{35} \tilde{A} C \tilde{h}^{5/2} \frac{\cos \theta}{F^2} \frac{\partial \tilde{h}}{\partial \tilde{x}} + O(\varepsilon). \end{aligned} \quad (3.161)$$

We recognize the work associated to the basal friction in the first term of the right-hand side. We still need to find a suitable expression for the remaining terms. First, it follows from the definition of β (3.98), the expression of $\tilde{U}^{(0)}$ (3.74) and Equation (3.83) that

$$\begin{aligned} & \frac{13\beta - 11}{250} \tilde{A}^3 \tilde{h}^{9/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} + \frac{2\lambda \cos \theta}{35 F^2} \tilde{A} \frac{\partial \tilde{h}}{\partial \tilde{x}} \tilde{h}^{5/2} \\ &= -\frac{3\beta}{250} \tilde{A}^3 \tilde{h}^{9/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} + \lambda \frac{10 \cos \theta}{\tilde{A}^2 B F^2} \tilde{U}^{(0)} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\tilde{\psi}^{(1)} - \frac{\tilde{A} \tilde{U}^{(1)}}{5 \tilde{h}^{1/2}} \right). \end{aligned} \quad (3.162)$$

By using equations (3.74), (3.78) and (3.83), we get the following exact equality:

$$-\frac{3}{250} \tilde{A}^3 \tilde{h}^{9/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} = \frac{\cos \theta}{F^2} \tilde{U}^{(0)} \left(\frac{\tilde{\alpha}_1(\theta)}{\tilde{h}^{1/2}} \tilde{U}^{(1)} + \tilde{\alpha}_2(\theta) \left(\tilde{\psi}^{(1)} - \frac{\tilde{A} \tilde{U}^{(1)}}{5 \tilde{h}^{1/2}} \right) \right), \quad (3.163)$$

where

$$\alpha_1(\theta) = -\frac{33C}{2AB(34C + 28)}, \quad \tilde{\alpha}_1(\theta) = -\frac{33C}{2\tilde{A}B(34C + 28)}, \quad (3.164)$$

and

$$\alpha_2(\theta) = \frac{77(9C - 12)}{2A^2B(34C + 28)}, \quad \tilde{\alpha}_2(\theta) = \frac{77(9C - 12)}{2\tilde{A}^2B(34C + 28)}. \quad (3.165)$$

Recall that $\tilde{U}^{(0)} = \tilde{U} + O(\varepsilon)$. Similarly, equation (3.73) can be written

$$\tilde{U}^{(1)} = \frac{\tilde{U} - \tilde{U}^{(0)}}{\varepsilon} + O(\varepsilon) = \frac{1}{\varepsilon} \left(\tilde{U} - \frac{2}{5} \tilde{A} \tilde{h}^{3/2} \right) + O(\varepsilon), \quad (3.166)$$

and similarly for (3.82)

$$\tilde{\psi}^{(1)} - \frac{\tilde{A} \tilde{U}^{(1)}}{5 \tilde{h}^{1/2}} = \frac{1}{\varepsilon} \left(\tilde{\psi} - \frac{\tilde{U}^2}{4 \tilde{h}^2} \right) + O(\varepsilon). \quad (3.167)$$

We can thus modify (3.163) and obtain

$$-\frac{3}{250} \tilde{A}^3 \tilde{h}^{9/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} = \frac{\tilde{U} \cos \theta}{\varepsilon F^2} \left[\frac{\tilde{\alpha}_1(\theta)}{\tilde{h}^{1/2}} \left(\tilde{U} - \frac{2}{5} \tilde{A} \tilde{h}^{3/2} \right) + \tilde{\alpha}_2(\theta) \left(\tilde{\psi} - \frac{\tilde{U}^2}{4 \tilde{h}^2} \right) \right] + O(\varepsilon). \quad (3.168)$$

In a similar fashion, we have

$$\lambda \frac{10}{\tilde{A}^2 B} \frac{\cos \theta}{F^2} \tilde{U}^{(0)} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\tilde{\psi}^{(1)} - \frac{\tilde{A} \tilde{U}^{(1)}}{5 \tilde{h}^{1/2}} \right) = \lambda \tilde{U} \frac{10 \cos \theta}{\varepsilon F^2 \tilde{A}^2 B} \left(\tilde{\psi} - \frac{\tilde{U}^2}{4 \tilde{h}^2} \right) + O(\varepsilon). \quad (3.169)$$

8.2. Expression of the critical value of λ

The dispersion relation of System (3.116) can be written $\det(M(\varepsilon)) = 0$, with $M(\varepsilon) = R \cos \theta / F^2 + i \varepsilon L$,

$$L = \begin{pmatrix} k - \omega & k & 0 \\ [(1 - \frac{3C}{4} + \frac{C\lambda}{7}) \frac{1}{F^2} + \frac{33}{64} + \frac{15\lambda}{176}] k & k - \omega & (\frac{11}{16} + \frac{5\lambda}{44}) k \\ 0 & 0 & k - \omega \end{pmatrix}, \quad (3.170)$$

and

$$R = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{3}{2B} - \frac{4\lambda}{5B} & \frac{1}{B} + \frac{4\lambda}{5B} & -\frac{8\lambda}{5B} \\ 3\alpha_1(\theta) - \alpha_2(\theta) & -2\alpha_1(\theta) + \alpha_2(\theta) & -2\alpha_2(\theta) \end{pmatrix}. \quad (3.171)$$

We denote by $g(\varepsilon \omega, \varepsilon k, F)$ the determinant $\det(M(\varepsilon))$. From the dispersion relation, k can be expressed as a function of (ω, F) :

$$g(\omega, k, F) = 0 \Leftrightarrow k = k(\omega, F). \quad (3.172)$$

We are now interested by the shape of the neutral stability curve in the long-wave limit, i.e. when ε goes to zero. This amounts to study the behaviour of $g(\omega, k, F)$ near $\omega = 0, k = 0, F = F_c$. We thus write a Taylor expansion of the dispersion relation at this point, up to order 4:

$$g(\omega, k, F) = \sum_{\ell=0}^4 \sum_{\ell_1+\ell_2+\ell_3=\ell} \frac{\omega^{\ell_1} k^{\ell_2} (F - F_c)^{\ell_3}}{(\ell_1!)(\ell_2!)(\ell_3!)} \frac{\partial^\ell g}{\partial k^{\ell_1} \partial \omega^{\ell_2} \partial F^{\ell_3}}(0, 0, F_c) + O(\|(\omega, k, F - F_c)\|^5). \quad (3.173)$$

A similar expansion can be written for the wave number k :

$$k(\omega, F) = \sum_{\ell=0}^4 \sum_{\ell_1+\ell_2=\ell} \frac{\omega^{\ell_1} (F - F_c)^{\ell_2}}{(\ell_1!)(\ell_2!)} \frac{\partial^\ell k}{\partial \omega^{\ell_1} \partial F^{\ell_2}}(0, 0, F_c) + O(\|(\omega, F - F_c)\|^5). \quad (3.174)$$

The expansions (3.173), (3.174) can be combined to obtain an expansion of $g(\omega, k(\omega, F), F)$:

$$g(\omega, k(\omega, F), F) = \sum_{\ell=0}^4 \sum_{\ell_1+\ell_2=\ell} \alpha_{\ell_1, \ell_2} \omega^{\ell_1} (F - F_c)^{\ell_2} + O(\|(\omega, F - F_c)\|^5), \quad (3.175)$$

where the coefficients α_{ℓ_1, ℓ_2} depend on the partial derivatives of g and k evaluated at $(\omega, k, F) = (0, 0, F_c)$. Since $g(\omega, k(\omega, F), F) = 0$, all the coefficients α_{ℓ_1, ℓ_2} are equal to 0. The relations $\alpha_{\ell_1, \ell_2} = 0$ then enable to compute explicitly all the partial derivatives of

3. Modèle consistant d'écoulement granulaire incompressible – 8. Appendix

k . After some computations, we obtain that the Taylor expansion of $\text{Im}(k)(\omega, F)$ is

$$\text{Im}(k) = -\frac{BF_c}{5}\omega^2(F - F_c) + a\omega^4 - \frac{B}{10}\omega^2(F - F_c)^2 + O(\|(\omega, F - F_c)\|^5). \quad (3.176)$$

where

$$a = \frac{B^3 F_c^4 (14 + 17C) [8890 - 2999C + 36\lambda(42 + 41C)] [847(3C - 4) + 20\lambda(84 + 157C)]}{559020000(28 - 21C + 4C\lambda)^2}. \quad (3.177)$$

The neutral stability curve is given by $\text{Im}(k)(\omega, F) = 0$. We deduce that this curve can also be written as $F_c = F_c(\omega)$, where $\partial F_c / \partial \omega|_{\omega=0} = 0$ and

$$\left. \frac{\partial^2 F_c}{\partial \omega^2} \right|_{\omega=0} = \frac{10a}{BF_c}. \quad (3.178)$$

It follows that $\partial^2 F_c / \partial \omega^2|_{\omega=0}$ vanishes when a vanishes, i.e. for the following critical values of λ :

$$\lambda_c^- = \frac{2999C - 8890}{36(42 + 41C)} < 0 \quad \text{and} \quad \lambda_c^+ = \frac{847(4 - 3C)}{20(84 + 157C)} > 0. \quad (3.179)$$

In order to have a well-posed model, it is required to have $\lambda \geq 0$ (otherwise U may blow up), and thus to discard λ_c^- . The only possible value is thus λ_c^+ .

8.3. Critical Froude number for first-order two-equation hyperbolic systems

We consider a very general first-order hyperbolic system of two equations for mass and momentum balance, modelling a fluid down an inclined plane. We assume that the pressure is hydrostatic and that the only other effects are the weight and a friction depending on the fluid depth h and of the average velocity U . Otherwise the rheology is not specified. It can be for instance the Newtonian rheology or the $\mu(I)$ granular rheology. The shape factor is equal to 1. Once written in a dimensionless form and linearized around the equilibrium values $h = 1$ and $U = 1$, the linear system for the small perturbations can be written with the matrix

$$\begin{pmatrix} k - \omega & k \\ ik \left(c_1 \frac{\cos \theta}{F^2} + c_2 \right) - \frac{c_3(\theta)}{F^2} & i(k - \omega) + \frac{c_4(\theta)}{F^2} \end{pmatrix}, \quad (3.180)$$

where c_1 , c_2 , $c_3(\theta)$ and $c_4(\theta)$ are constants, c_3 and c_4 depending on the angle θ . We assume that the wave number k is real and the angular frequency ω is complex. The neutral stability is then defined by $\text{Im}(\omega) = 0$. Writing that the determinant of the

3. Modèle consistant d'écoulement granulaire incompressible – 8. Appendix

above matrix is equal to zero yields, in the neutral stability conditions,

$$\omega = k \left(1 + \frac{c_3}{c_4} \right) \quad (3.181)$$

which means that the phase velocity is constant on the neutral stability curve, and

$$F_c = c_4 \frac{\sqrt{c_1 \cos \theta}}{\sqrt{c_3^2 - c_2 c_4^2}} \quad (3.182)$$

where F_c is the critical Froude number. The critical Froude number is therefore independent of the frequency.

For a consistent first-order two-equation model for the Newtonian rheology, the constants are (see Richard, Gislou, et al. 2019)

$$c_2 = 4 - \frac{18}{5} c_1; \quad c_3 = 2c_1 \sin \theta; \quad c_4 = c_1 \sin \theta, \quad (3.183)$$

and c_1 can be chosen arbitrarily. The dimensionless phase velocity is then equal to 3 and the Froude number is

$$F_c = \sqrt{\frac{18}{5} \cos \theta}, \quad \mathcal{F}_c = \sqrt{\frac{18}{5}}, \quad (3.184)$$

which are the consistent values of the phase velocity and of the critical Froude number for the Newtonian rheology in the long-wave limit.

For the model of Forterre and Pouliquen 2003 or Gray and Edwards 2014 without diffusion, we have

$$c_1 = 1; \quad c_2 = 0; \quad c_3 = \frac{3 \cos \theta}{2B}; \quad c_4 = \frac{\cos \theta}{B}. \quad (3.185)$$

This gives a phase velocity equal to 5/2 and a critical Froude number

$$F_c = \frac{2}{3} \sqrt{\cos \theta}, \quad \mathcal{F}_c = \frac{2}{3}, \quad (3.186)$$

which is different from the consistent critical Froude number (3.133) for the $\mu(I)$ rheology in the long-wave limit (Forterre 2006).

8.4. Diffusive terms in the energy balance equation

Equation (3.48) yields

$$\tilde{u}^{(0)} \frac{\partial \tilde{\tau}_{xx}^{(0)}}{\partial \tilde{x}} = \frac{2\tilde{A}}{3} (\tilde{h}^{3/2} - (\tilde{h} - \tilde{z})^{3/2}) \frac{\partial \tilde{\tau}_{xx}^{(0)}}{\partial \tilde{x}}. \quad (3.187)$$

3. Modèle consistant d'écoulement granulaire incompressible – 8. Appendix

From previous computations, we obtain

$$\int_0^{\tilde{h}} \tilde{h}^{3/2} \frac{\partial \tilde{\tau}_{xx}^{(0)}}{\partial \tilde{x}} = \frac{2}{3} \phi \sin \theta \left[\left(\frac{\partial \tilde{h}}{\partial \tilde{x}} \right)^2 \tilde{h}^{5/2} + \frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} \frac{\tilde{h}^{7/2}}{2} \right] \quad (3.188)$$

We also compute with (3.136)

$$\int_0^{\tilde{h}} (\tilde{h} - \tilde{z})^{3/2} \frac{\partial \tilde{\tau}_{xx}^{(0)}}{\partial \tilde{x}} = \frac{2}{3} \phi \sin \theta \left[\left(\frac{\partial \tilde{h}}{\partial \tilde{x}} \right)^2 \frac{\tilde{h}^{5/2}}{20} + \frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} \frac{\tilde{h}^{7/2}}{7} \right] \quad (3.189)$$

Hence

$$\int_0^{\tilde{h}} \tilde{u}^{(0)} \frac{\partial \tilde{\tau}_{xx}^{(0)}}{\partial \tilde{x}} = \frac{2}{9} \tilde{A} \phi \sin \theta \left[\left(\frac{\partial \tilde{h}}{\partial \tilde{x}} \right)^2 \frac{19}{10} \tilde{h}^{5/2} + \frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} \frac{5}{7} \tilde{h}^{7/2} \right] \quad (3.190)$$

and

$$\begin{aligned} \int_0^{\tilde{h}} \tilde{u}^{(0)} \frac{\partial \tilde{\tau}_{xx}^{(0)}}{\partial \tilde{x}} - \phi F^2 \frac{\partial}{\partial \tilde{x}} \left(\tilde{v}_{\text{eff}} \tilde{h} \tilde{U}^{(0)} \frac{\partial \tilde{U}^{(0)}}{\partial \tilde{x}} \right) &= \frac{2 \tilde{A} \phi \sin \theta}{3} \left[-\frac{1}{15} \tilde{h}^{5/2} \left(\frac{\partial \tilde{h}}{\partial \tilde{x}} \right)^2 + \frac{4}{105} \tilde{h}^{7/2} \frac{\partial^2 \tilde{h}}{\partial \tilde{x}^2} \right] \\ &= \phi F^2 \left[\frac{8}{7} \frac{\partial}{\partial \tilde{x}} \left(\tilde{v}_{\text{eff}} \tilde{h}^3 \frac{\partial \tilde{\psi}^{(0)}}{\partial \tilde{x}} \right) - \frac{2}{3} \tilde{v}_{\text{eff}} \tilde{h} \left(\frac{\partial \tilde{U}^{(0)}}{\partial \tilde{x}} \right)^2 \right]. \end{aligned} \quad (3.191)$$

4. Régularisation et validation d'un modèle moyenné sur la profondeur pour les écoulements granulaires consistant avec la rhéologie $\mu(I)$ incompressible

Dans ce chapitre, on présente une version régularisée du modèle introduit au chapitre 3. En effet, le modèle du chapitre 3 a été dérivé en effectuant un développement asymptotique au voisinage d'un écoulement stationnaire et uniforme. Par conséquent, il n'est valide que dans un intervalle de pentes restreint pour lequel ces écoulements existent. En dehors de cet intervalle, et notamment lors des transitions liquide/solide et liquide/gaz, le modèle écrit au chapitre 3 ne peut pas être utilisé tel quel. Afin d'obtenir un modèle qui reste pertinent pour des pentes arbitraires, on introduit dans ce chapitre un coefficient de régularisation permettant de modéliser une transition entre différents régimes d'écoulement. Ce coefficient est écrit de manière à préserver la consistance à l'ordre 1 avec la rhéologie $\mu(I)$ dans l'intervalle de pentes pour lesquelles il existe des écoulements stationnaires et uniformes. En dehors de ces pentes, le modèle garde une structure bien posée. Les prédictions du modèle régularisé sont ensuite validées par des comparaisons avec des données expérimentales dans une grande gamme de régimes non stationnaires et non uniformes. On étudie tout d'abord le cas des fronts secs, et on montre que le modèle prédit avec succès la variation des profils avec le nombre de Froude, tout en gardant un front bien défini. On montre ensuite que le modèle donne des résultats en bon accord avec des données expérimentales dans le cas de l'effondrement d'une colonne de grains sur une pente faible. Enfin, on effectue une troisième validation dans une configuration avec une pente importante et une topographie variable. Ce chapitre est l'objet d'un projet d'article en cours de rédaction.

1. Introduction

Granular flow models are used in a wide scope of applications, from the prediction of natural hazards (dry snow avalanches, landslides) to the description of many industrial

4. Régularisation et validation du modèle d'écoulement granulaire incompressible – 1. Introduction

processes. In the dense regime, the behaviour of these flows is mainly driven by the mechanical interactions between the particles (friction and collisions). At the flow scale, collective effects can emerge from these local interactions, increasing the complexity of the description of the media. For a general introduction to the subject, we refer to the book of Andreotti et al. [2013](#).

In the context of gravity flows down a slope, an efficient way to model shallow flows is to use depth-averaged equations. Indeed, depth-averaged models have a reduced computation time due to space dimension reduction and the incorporation of boundary conditions into the depth-averaged variables. A depth-averaged two-equation model specifically written for the modelling of granular flows was proposed by Savage et al. [1989](#). The model used shallow water equations, complemented with source terms accounting for a Coulomb friction law. More complex versions of this system were also implemented in order to take into account two dimensions, an arbitrary topography, and were used in many geophysical applications (Denlinger et al. [2004](#); Pudasaini et al. [2003](#); Bouchut and Westdickenberg [2004](#); Mangeney-Castelnau, Bouchut, et al. [2005](#)).

In the work of Pouliquen [1999b](#), stationary and uniform granular flows were obtained on a rough bottom. In particular, it was shown that for the same material, the equilibrium could be attained on different slopes. As this behaviour is not permitted by a constant friction coefficient, a different friction law was proposed in the same paper. The friction coefficient is then a function of the height of the flow h and of the mean velocity U . This function is determined by the mean of an empirical coefficient representing the depth of the granular deposit left by a stationary uniform flow, called h_{stop} . This law enabled to predict accurately the shape of dry granular fronts (Pouliquen [1999a](#)). A more complex version was then proposed by Pouliquen and Forterre [2002](#), which enabled the transition towards a static friction coefficient close to flow stopping. The static friction coefficient was written with the help of the empirical parameter h_{start} , representing the smallest height at which a granular media initially at rest starts to flow. This resulted in a non-monotonic friction law, enabling to take hysteresis into account and was successfully applied to the spreading of a granular mass.

This law was then generalised into a local rheology (GDR MiDi [2004](#)), by the assumption that the friction law of Pouliquen [1999b](#) was valid in every point of the flow. To obtain a local law, the dependence on the global variables h and U was replaced by a dependence on a local dimensionless parameter I called inertial number, depending on the shear rate and on the pressure. This local law was then written in a tensorial form by Jop et al. [2006](#), giving rise to a constitutive law called $\mu(I)$ -rheology. This rheology enabled to give a precise description of stationary uniform flows down inclined planes. In particular, it predicts a Bagnold velocity profile, which is consistent with experimental observation and discrete simulations (GDR MiDi [2004](#)). This tensorial rheology was used to study granular flows in various regimes (see e.g. Lagr  e, Staron, et al. [2011](#); Ionescu et al. [2015](#); H. A. Martin et al. [2023](#)). However, Barker, Schaeffer, Boh  rquez, et al. [2015](#) showed that this rheology is ill-posed in the limits of low and high inertial numbers, exhibiting Hadamard's instability. Some generalisations of

4. Régularisation et validation du modèle d'écoulement granulaire incompressible – 1. Introduction

the rheology were proposed in order to regularise this behaviour, based on either a modification of the friction law (Barker and Gray 2017), or on the integration of compressibility effects into the rheology (Heyman et al. 2017; Barker, Schaeffer, Shearer, et al. 2017; Schaeffer et al. 2019).

The $\mu(I)$ -rheology was also used to improve the depth-averaged models. Granular roll waves were observed by Forterre and Pouliquen 2003 above a critical Froude number. The model of Pouliquen 1999b was used to estimate this critical Froude number, with a good agreement with the observations, although the theoretical prediction overestimated slightly the measured threshold. However, a cut-off frequency above which no instability developed was observed, in contradiction with the predictions of the model. Forterre 2006 and Gray and Edwards 2014 added a depth-averaged diffusive terms to the two-equation model of Pouliquen 1999b with friction law. They showed that this enables to predict accurately the cut-off frequency of the roll-wave instability observed by Forterre and Pouliquen 2003, without affecting the instability threshold given by the critical Froude number. The model of Gray and Edwards 2014 was used to predict erosion-deposition waves (Edwards and Gray 2015), and the formation of levees (Edwards, Viroulet, et al. 2017). The Bagnold profile was also taken into account via the addition of a shape factor of 5/4 into the depth-averaged momentum equation. Saingier et al. 2016 showed that this enables to predict the flattening of the front profile with the increase of the Froude number, with a very good accuracy, and that the model with a shape factor of 5/4 is much better when the Froude number is above 1. However, this addition of a shape factor different from 1 resulted in a nonphysical behaviour: an infinite precursor film appears, which is not observed in the experiments. Furthermore, taking into account a shape factor of 5/4 in the model of Pouliquen 1999b yields a worse prediction of the instability threshold of the roll waves instability.

An alternative method to take into account non uniform velocity profiles in depth-averaged models was proposed by Teshukov 2007. This method consists in considering an augmented model, with additional evolution equations for quantities accounting for the velocity fluctuations, obtained consistently with the depth-averaging procedure. This method yields a model that admits an energy equation, which is not the case for a two-equation model with a shape factor different from 1 (Richard and Gavriluk 2012). In the one dimensional case, only one additional variable is needed, and it is called enstrophy because it is related to the square of the vorticity (Richard and Gavriluk 2012). This method was used to derive depth-averaged models for Newtonian fluids (Richard, Ruyer-Quil, et al. 2016; Kazakova et al. 2019) and viscoplastic flows (Denisenko, Richard, et al. 2023).

This method was implemented by the authors in the case of granular flows (Deleage and Richard 2025). The three-equation model obtained was also shown to be consistent at order one with the $\mu(I)$ -rheology. This means that a solution of the $\mu(I)$ constitutive equations yields an approximate solution of the depth-averaged model, up to an $O(\varepsilon^2)$ error. Here, ε denotes the usual shallow-water parameter. This consistency was obtained by taking into account the first-order correction to the friction law of Pouliquen 1999b, as well as the Bagnold profile through the enstrophy variable.

A consequence of the consistency is that the instability threshold of the roll waves instability is the same for the depth-averaged model and for the $\mu(I)$ -rheology in the long-wave limit, which was computed explicitly by Forterre 2006. This is not the case with the two-equation models. The three-equation model has a hyperbolic structure with source terms, compatible with an energy equation. A higher-order version of the model was also proposed, with diffusive terms similar to the one of Gray and Edwards 2014. This version of the model predicts accurately the phase velocity and the cut off frequency of the roll wave instability.

The limits of the model of Deleage and Richard 2025 were studied by the authors. They showed that the well-posedness of the model does not hold in the liquid/solid and liquid/gas transitions, i.e. when the slope reaches a lower or an upper threshold. This is a consequence of the fact that the model was written using an asymptotic expansion valid only close to stationary and uniform flows. For the $\mu(I)$ -rheology, the range of slopes at which these flows can be observed is limited, and the three-equation model of Deleage and Richard 2025 cannot be used outside this range. In this paper, we propose a regularised version of the three-equation model of Deleage and Richard 2025, valid at any slope. We then show by comparisons with experimental data that the regularised model is accurate in a lot of different regimes. In section 2, we present in details the regularised model. In section 3, we study the dry fronts given by the regularised model. The regime of small slopes at which the flow stops is analysed in section 4. Finally, the performances of the model on a variable topography, enabling a transition from an accelerated regime on a steep slope to a stopping phase on a horizontal bottom, are investigated in section 5.

2. Presentation of the model

2.1. A brief recall of the model of Deleage and Richard 2025

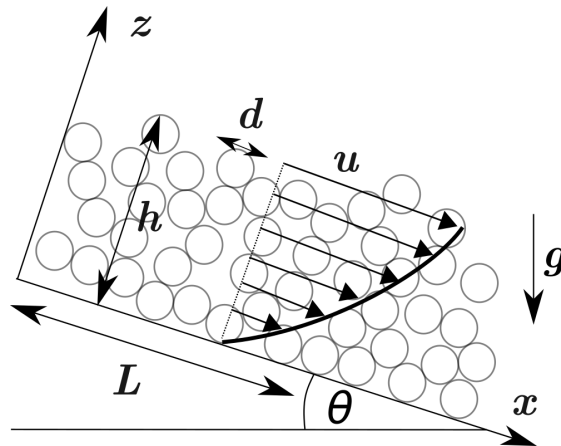


Figure 4.1. – Definition sketch

4. Régularisation et validation du modèle d'écoulement granulaire incompressible –
2. Presentation of the model

The model of Deleage and Richard 2025 is a three-equation model for a granular flow down an inclined plane of angle θ under the action of gravity g (see figure 4.1). The grain diameter d , the particles density ρ_p and the solid volume fraction ϕ are supposed constant. The model is derived consistently from the incompressible $\mu(I)$ -rheology, defined by the three parameters μ_1, μ_2 and I_m , such that the internal friction coefficient μ is given as a function of the inertial number I as (Jop et al. 2006):

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + \frac{I_m}{I}}, \quad \text{with} \quad I = \frac{|\dot{\gamma}|d}{\sqrt{p/\rho_p}}. \quad (4.1)$$

The scalars $|\dot{\gamma}|$ and p respectively stand for the local shear and pressure inside the flow.

The three variables of the model are the flow depth h , the mean velocity U and the enstrophy ψ , which is linked to the shear (see the definition in Deleage and Richard 2025). Their evolution is governed by the following system:

$$\left\{ \begin{array}{l} \frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0, \end{array} \right. \quad (4.2a)$$

$$\left\{ \begin{array}{l} \frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left(hU^2 + \beta h^3 \psi + K \frac{gh^2}{2} \cos \theta \right) \\ = g \cos \theta \left[h(\tan \theta - \mu_b) + \frac{10\lambda}{A^2 B} \left(\psi - \frac{U^2}{4h^2} \right) \right], \end{array} \right. \quad (4.2b)$$

$$\left\{ \begin{array}{l} \frac{\partial h\psi}{\partial t} + \frac{\partial hU\psi}{\partial x} = \frac{2gU \cos \theta}{h^2} \left[\frac{\alpha_1}{h^{1/2}} \left(U - \frac{2}{5} Ah^{3/2} \right) + \alpha_2 \left(\psi - \frac{U^2}{4h^2} \right) \right]. \end{array} \right. \quad (4.2c)$$

Equation (4.2a) is the depth-averaged mass equation, equation (4.2b) is the depth-averaged horizontal momentum equation, and equation (4.2c) is the enstrophy equation. These three equations are written in conservative form. The expressions of the coefficients A, B, C, α_1 and α_2 are

$$A = I_m \frac{\tan \theta - \mu_1}{\mu_2 - \tan \theta} \frac{\sqrt{\phi g \cos \theta}}{d}, \quad B = \frac{\mu_2 - \mu_1}{(\mu_2 - \tan \theta)(\tan \theta - \mu_1)}, \quad (4.3)$$

$$C = \tan \theta \left(2 \tan \theta - \frac{1}{B} \right), \quad (4.4)$$

$$\alpha_1 = -\frac{33C}{2AB(34C + 28)}, \quad \alpha_2 = \frac{77(9C - 12)}{2A^2B(34C + 28)}. \quad (4.5)$$

The parameters λ, β and K are given by

$$\lambda = \frac{847(4 - 3C)}{20(157C + 84)}, \quad \beta = \frac{11}{16} + \frac{5\lambda}{44}, \quad K = 1 - \frac{3C}{4} + \frac{\lambda C}{7}, \quad (4.6)$$

and the basal friction μ_b is given by

$$\mu_b = \mu_1 + \frac{\mu_2 - \mu_1}{I_m \frac{2h^{3/2} \sqrt{\phi g \cos \theta}}{5dU} + 1}. \quad (4.7)$$

These equations form a hyperbolic system of conservation laws with relaxation source terms. They are compatible with the following energy balance equation:

$$\begin{aligned} \frac{\partial h e}{\partial t} + \frac{\partial}{\partial x} (h U e + \Pi U) &= g U \cos \theta \\ &\times \left[h(\tan \theta - \mu_b) + \frac{\beta \alpha_1}{h^{1/2}} \left(U - \frac{2}{5} A h^{3/2} \right) + \left(\beta \alpha_2 + \frac{10\lambda}{A^2 B} \right) \left(\psi - \frac{U^2}{4h^2} \right) \right], \end{aligned} \quad (4.8)$$

where the specific energy is

$$e = \frac{U^2}{2} + \frac{\beta h^2 \psi}{2} + K \frac{g h}{2} \cos \theta \quad (4.9)$$

and

$$\Pi = \beta h^3 \psi + K \frac{g h^2}{2} \cos \theta. \quad (4.10)$$

As it was shown in Deleage and Richard 2025, this model enables to recover the threshold of the $\mu(I)$ -rheology computed by Forterre 2006 for the granular roll waves instability, taking into account a steady uniform flow with a Bagnold profile. However, the $\mu(I)$ -rheology only allows the existence of such steady uniform flows for an angle θ such that

$$\mu_1 < \tan \theta < \mu_2. \quad (4.11)$$

Indeed, these flows can only occur when the weight is compensated by the friction, which gives the equilibrium condition $\mu = \tan \theta$. For the standard form of the $\mu(I)$ curve (4.1), the friction coefficient μ is restricted to the interval (μ_1, μ_2) . Hence no steady uniform flow can be obtained when the condition (4.11) is not satisfied (apart from uniform flow at rest). As a consequence, the definitions (4.3), (4.4) (4.5), (4.6) of the coefficients of the model are only valid when condition (4.11) holds true. For an inclination angle out of the range $(\arctan \mu_1, \arctan \mu_2)$, the model (4.2) cannot be used. We refer to Deleage and Richard 2025 for more details concerning these limitations of system (4.2).

Another problem of system (4.2), which is also inherited from the $\mu(I)$ rheology, can occur even when the condition (4.11) is satisfied, and is linked with the value of the parameter μ_2 . In Forterre 2006, Forterre showed that, for the $\mu(I)$ -rheology, the roll waves instability occurs when the Froude number $\mathcal{F}$ is greater than a critical Froude number $\mathcal{F}_c$ given by

$$\mathcal{F}_c^2 = \frac{16}{25} (4 - 3C), \quad (4.12)$$

and C is given by equation (4.4). In Deleage and Richard 2025, the authors showed that if $\mu_2 > \sqrt{2/3}$, there are some values of θ such that (4.11) is satisfied and $4 - 3C < 0$,

which implies that every steady uniform flow is unstable, whatever the Froude number is (see (4.12)). In system (4.2), this unstable behaviour is materialised by the fact that the coefficients K and α_2 change sign, leading to a model that can be ill-posed. In the next subsection, we propose a regularised version of system (4.2), which is defined and well-posed for every inclination angle θ .

2.2. Regularised model

In order to regularise system (4.2), we introduce the coefficient f defined by

$$f = \begin{cases} \exp\left(2 - \frac{25U^2}{4A^2h^3} - \frac{4A^2h^3}{25U^2}\right), & \mu_1 < \tan\theta < \mu_2, \\ 0, & \text{else.} \end{cases} \quad (4.13)$$

Let us motivate this formula. First, it follows from (4.13) that $0 \leq f \leq 1$, and that $f = 0$ in the following three cases:

- when (4.11) is not satisfied;
- when $h = 0$, i.e. in a dry front;
- when $U = 0$, i.e. when the flow stops.

Note that, for a steady uniform flow, the depth-averaged velocity and the depth are linked via the relation $U = 2Ah^{3/2}/5$ (Pouliquen 1999b, see also Deleage and Richard 2025), hence $f = 1$. The coefficient f may thus be seen as a quantification of the distance between the solution of system (4.2) and a steady uniform flow.

As a consequence, we seek to write the regularised model as an interpolation between system (4.2) when $f = 1$ and another system when $f = 0$. When doing so, one must ensure that the consistency between system (4.2) and the $\mu(I)$ -rheology is preserved by the regularisation procedure. Indeed, let us introduce as usual h_0 the typical height of the flow, and L_0 a typical horizontal length. The shallow-water parameter ε is defined by

$$\varepsilon = \frac{h_0}{L_0}. \quad (4.14)$$

Deleage and Richard 2025 showed that system (4.2) is consistent at order 1 with the $\mu(I)$ -rheology, which means that an exact solution of the governing equations of the $\mu(I)$ -rheology yields an approximate solution of system (4.2), up to an $O(\varepsilon^2)$ remainder. Let us now suppose that condition (4.11) is satisfied. We introduce the dimensionless variables $\tilde{h}$ and $\tilde{U}$ given by

$$\tilde{h} = \frac{h}{h_0}, \quad \tilde{U} = \frac{5U}{2Ah_0^{3/2}}. \quad (4.15)$$

It follows that the coefficient f can be written

$$f = \exp\left(2 - \frac{\tilde{U}^2}{\tilde{h}^3} - \frac{\tilde{h}^3}{\tilde{U}^2}\right). \quad (4.16)$$

Then if we assume as in Deleage and Richard 2025 that $\tilde{U}$ satisfies an asymptotic

4. Régularisation et validation du modèle d'écoulement granulaire incompressible – 2. Presentation of the model

expansion, given by

$$\tilde{U} = \tilde{U}^{(0)} + \varepsilon \tilde{U}^{(1)} + O(\varepsilon^2) = \tilde{h}^{3/2} + \varepsilon \tilde{U}^{(1)} + O(\varepsilon^2), \quad (4.17)$$

we deduce that

$$f = \exp [O(\varepsilon^2)] = 1 + O(\varepsilon^2). \quad (4.18)$$

Thus, by using carefully the coefficient f , we can ensure that the error terms are at most $O(\varepsilon^2)$, and that the consistency of the regularised model is preserved.

It remains to specify the system to be used far from steady uniform flows, i.e. when $f = 0$. In order to do this, we treat each equation of (4.2) separately.

1. Equation (4.2a) is the equation of conservation of mass. It is expected to be valid in every flow regime, so we keep it unchanged in the regularised version of the model.
2. Equation (4.2b) is the balance of momentum. We do not modify the left-hand side of the equation in the regularised version, but we change the expression of the right-hand side:
 - We replace the basal friction μ_b by the non-monotonic friction law denoted here $\mu(h, U)$ that was proposed by Pouliquen and Forterre 2002 and given by

$$\mu(h, U) = \begin{cases} \mu_b, & \frac{|U|}{\sqrt{gh \cos \theta}} > \beta_*, \\ \left(\frac{|U|}{\beta_* \sqrt{gh \cos \theta}} \right)^\gamma (\mu_1 - \mu_3) + \mu_3 + \frac{\mu_2 - \mu_1}{\frac{h}{L} + 1}, & 0 < \frac{|U|}{\sqrt{gh \cos \theta}} < \beta_*, \\ \min \left(\mu_3 + \frac{\mu_2 - \mu_1}{\frac{h}{L} + 1}, \left| \tan \theta - K \frac{\partial h}{\partial x} \right| \right), & U = 0, \end{cases} \quad (4.19)$$

Here the constant β_* is a parameter appearing in the scaling

$$\frac{|U|}{\sqrt{gh \cos \theta}} = \mathcal{F} = \beta_* \frac{h}{h_{stop}(\theta)} \quad (4.20)$$

(with $h_{stop}(\theta)$ the depth of the static layer left by a steady uniform flow, see Pouliquen 1999b) and characterising the transition between the static and dynamic regimes. Other criteria for the transition were proposed (see Edwards, Viroulet, et al. 2017; Edwards, Russell, et al. 2019) but we choose to keep here the original formulation of Pouliquen and Forterre 2002. An advantage of this non-monotonic law is that it allows to capture frictional hysteresis. A drawback is that it involves, in addition to the three parameters μ_1, μ_2, I_m of the rheology, four other parameters $(\mu_3, \beta_*, L, \gamma)$ which are not always determined by experimental measurements. In order to be able to compare the performances of our model with laboratory data even when these parameters are not specified, we proceed as in Poulain et al. 2023 and set in this case $\beta_* = 0.136$, $L = 1.3d$, $\gamma = 10^{-3}$ and $\mu_3 = \tan \theta_3$, where

4. Régularisation et validation du modèle d'écoulement granulaire incompressible –
2. Presentation of the model

$$\theta_3 = \theta_1 + 1.2^\circ \text{ and } \tan \theta_1 = \mu_1.$$

- The relaxation term with a factor λ in the momentum equation ensures consistency in order to obtain the right instability threshold near a steady uniform flow. When condition (4.11) is not true, there is no purpose to keep this term. We thus multiply this term by the coefficient f and set it equal to zero when $\tan \theta$ is outside (μ_1, μ_2) .
- 3. Equation (4.2c) is the equation of enstrophy. The left-hand side does not need to be regularised since it is only a conservation equation. However, the right-hand side is made of relaxation terms ensuring consistency near stationary and uniform flows. As in the momentum equation, we choose to multiply these terms by the factor f in order to set them equal to zero when $f = 0$. Recall that at equilibrium, the enstrophy ψ is equal to $U^2/(4h^2)$, corresponding to a Bagnold velocity profile, and to a shape factor $\alpha = 5/4$ (the shape factor can be defined with the help of the enstrophy by $\alpha = 1 + h^2\psi/U^2$). Far from equilibrium, few data on the velocity profile or on the shape factor of a granular flow are available. However, Lagrée, Saingier, et al. 2017 showed that, in a dry front, the shape factor is equal to 5/4, except close to the front where it goes to 1 (see section 3). We thus choose to keep an enstrophy relaxing to $U^2/(4h^2)$ (i.e. a Bagnold profile) far from equilibrium, except close to a dry front where we want the enstrophy to relax towards 0. A way to do this is to impose the following enstrophy equation when $f = 0$:

$$\frac{\partial h\psi}{\partial t} + \frac{\partial hU\psi}{\partial x} = -\frac{1}{\tau} \left(\psi - e^{-\frac{5d}{h}} \frac{U^2}{4h^2} \right), \quad (4.21)$$

where τ is a relaxation time (in fact, τ has the dimension of a time divided by a length, since the time derivative is written on $h\psi$). Indeed, far from a dry front, $h \gg d$, hence $e^{-\frac{5d}{h}} \approx 1$ and the enstrophy relaxes towards $U^2/(4h^2)$ (Bagnold profile). Close to a dry front, $h \rightarrow 0$, hence $e^{-\frac{5d}{h}}$ goes to zero. We now need to specify the relaxation time τ . We cannot choose the same one as in equation (4.2c), since it depends on the coefficient α_2 which is not defined when $\tan \theta$ is not between μ_1 and μ_2 . However, we can use the equilibrium expression $U = 2Ah^{3/2}/5$ to write the relaxation time of equation (4.2c) as

$$-\frac{1}{\tau} = \frac{2gU\alpha_2 \cos \theta}{h^2} \approx \frac{4gA\alpha_2 \cos \theta}{5h^{1/2}} = \frac{4d}{5I_m} \sqrt{\frac{g \cos \theta}{\phi h}} \frac{77(9C - 12)}{2(34C + 28)} \frac{(\mu_2 - \tan \theta)^2}{\mu_2 - \mu_1} \quad (4.22)$$

We could keep this expression to define τ , but it vanishes when $\tan \theta = \mu_2$. In order to avoid this, we use the same expression, replacing the factor $(\mu_2 - \tan \theta)^2$ by $(\mu_2 - \mu_1)^2$.

Finally, we change the definition of the constant C in some cases in order to get rid of

the instability described at the end of §2.1. We thus set

$$C = \begin{cases} 2\mu_1^2 & \text{if } \tan \theta \leq \mu_1, \\ \min \left[2\mu_2^2, \tan \theta \left(2 \tan \theta - \frac{1}{B} \right) \right] & \text{if } \mu_1 < \tan \theta \text{ and } \mu_2 < \sqrt{\frac{2}{3}}, \\ \min \left[1, \tan \theta \left(2 \tan \theta - \frac{1}{B} \right) \right] & \text{if } \mu_1 < \tan \theta \text{ and } \mu_2 \geq \sqrt{\frac{2}{3}}. \end{cases} \quad (4.23)$$

The regularised system can be written:

$$\left\{ \begin{array}{l} \frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0, \end{array} \right. \quad (4.24a)$$

$$\left\{ \begin{array}{l} \frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left(hU^2 + \beta h^3 \psi + K \frac{g h^2}{2} \cos \theta \right) \\ = g \cos \theta \left[h[\tan \theta - \mu(h, U)] + \frac{10\lambda f}{A^2 B} \left(\psi - \frac{U^2}{4h^2} \right) \right], \end{array} \right. \quad (4.24b)$$

$$\left\{ \begin{array}{l} \frac{\partial h\psi}{\partial t} + \frac{\partial hU\psi}{\partial x} = \frac{2f^\delta g U \cos \theta}{h^2} \left[\frac{\alpha_1}{h^{1/2}} \left(U - \frac{2}{5} A h^{3/2} \right) + \alpha_2 \left(\psi - \frac{U^2}{4h^2} \right) \right] \\ + (1 - f^\delta) \frac{\alpha_3}{h^{1/2}} \left(\psi - e^{-\frac{5d}{h}} \frac{U^2}{4h^2} \right). \end{array} \right. \quad (4.24c)$$

The coefficients $A, B, \alpha_1, \alpha_2, \lambda, K, \beta$ are defined as before by the equations (4.3), (4.5), (4.6), with the regularised definition of C given in (4.23). The regularisation coefficient f is defined in equation (4.13). By convention, any term having the coefficient f in factor is set equal to zero when $f = 0$. The non-monotonic friction law is defined in (4.19), and the coefficient α_3 is equal to

$$\alpha_3 = \frac{4d}{5I_m} \sqrt{\frac{g \cos \theta}{\phi}} \frac{77(9C - 12)}{2(34C + 28)} (\mu_2 - \mu_1), \quad (4.25)$$

with C defined in (4.23). The only parameter which has not been discussed yet is the exponent δ in the regularised enstrophy equation (4.24c). This parameter is used to control the shape factor in a dry front and is set equal to $\delta = 1/20$ (see section 3).

The regularised model is still a hyperbolic system of conservation laws with source terms. It admits the energy equation

$$\begin{aligned} \frac{\partial h e}{\partial t} + \frac{\partial}{\partial x} (h U e + \Pi U) &= h^{3/2} (1 - f^\delta) \alpha_3 \left(\psi - e^{-\frac{5d}{h}} \frac{U^2}{4h^2} \right) + g U \cos \theta \\ &\times \left[h[\tan \theta - \mu(h, U)] + \frac{\beta f^\delta \alpha_1}{h^{1/2}} \left(U - \frac{2}{5} A h^{3/2} \right) + \left(\beta f^\delta \alpha_2 + \frac{10\lambda f}{A^2 B} \right) \left(\psi - \frac{U^2}{4h^2} \right) \right], \end{aligned} \quad (4.26)$$

and is well-posed for every inclination angle θ . When $\mu_1 < \tan \theta < \mu_2$, the fact that $f = 1 + O(\varepsilon^2)$ (see equation (4.18)) implies that the regularised model is fully consistent with the $\mu(I)$ -rheology up to order 1 (except in the special cases discussed hereafter).

Furthermore, one can check that the linearised version of (4.24) around a stationary uniform flow is exactly the same as for the non regularised system (4.2). As a consequence, the theoretical predictions of neutral stability curve, phase velocity and spatial growth rate are the same and the study of Deleage and Richard 2025 can be used for the regularised model.

When $\mu_1 < \tan\theta < \mu_2$, the consistency of the regularised system (4.24) with the $\mu(I)$ -rheology is lost in the two following cases:

- When the Froude number is below the threshold β_* , the friction law of the regularised model changes and transitions towards a static coefficient. This choice was used by Pouliquen and Forterre 2002 and enables to capture the deposit of height $h_{stop}(\theta)$ which is left by a steady uniform flow. Note that the $\mu(I)$ -rheology predicts that steady uniform flows can have an arbitrarily small height. This is in contradiction with experimental observations, as stated by Pouliquen and Forterre 2002: "no steady uniform flow is observed with a lower Froude number, i.e. with a thickness h less than h_{stop} ". The loss of consistency with the rheology is thus motivated by the fact that the predictions of the rheology are not in agreement with the observations in this regime.
- In the special case where $\mu_2 > \sqrt{2/3}$ and $\tan\theta$ is close to μ_2 , the regularised definition of C (see (4.23)) also implies that consistency with the rheology is lost. In this case, the rheology predicts that all steady uniform flows are unstable, and the non regularised system (4.2) can become ill-posed. Few knowledge of the behaviour of the flow in this regime is available (see however the work of Börzsönyi and Ecke 2006). In the absence of more information, we choose to set $C = 1$ in this case (see equation (4.23)) in order to keep a well-posed system, thus losing consistency with the rheology. Note that the slopes at which this problem can occur are quite restricted, since θ has to be smaller than $\theta_2 = \arctan(\mu_2)$ and greater than $\arctan(1/\sqrt{2}) \approx 35^\circ$ (the exact lower bound for θ is greater than 35° and depends on μ_1 and μ_2). Few experimental data in this regime are available, and we did not compare the performances of our regularised model in this case.

As explained before, the behaviour of the regularised model close to steady uniform flows is the same as the one of the non regularised model, which was studied in Deleage and Richard 2025. In the next sections, we perform numerical validations of the regularised model in three regimes far from steady uniform flows: dry fronts, flow stopping, accelerated flows. We compare the performances of system (4.24) with both experimental data and the classical two-equation model, given by

$$\begin{cases} \frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0, \\ \frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left(\alpha hU^2 + \frac{gh^2}{2} \cos\theta \right) = gh \cos\theta [\tan\theta - \mu(h, U)], \end{cases} \quad (4.27)$$

where $\mu(h, U)$ is defined by (4.19) and the shape factor α is equal to 1, except in section 3 where we also use the value $\alpha = 5/4$.

3. Study of dry fronts

Dry granular fronts are a simple example of flows that are not steady uniform. Indeed, they can be described as a travelling wave, i.e. as a solution propagating at a constant speed c (see Andreotti et al. 2013). As recalled hereafter, granular flow fronts are well-predicted by two-equation models, hence it is important to see also how the three-equation model (4.24) performs. In the case of the two-equation model (4.27), the mass conservation implies that the mean velocity U is constant in a dry front and equal to the propagation speed c (Andreotti et al. 2013). The shape of the front is thus described by the following ordinary differential equation (ODE) (Andreotti et al. 2013; Saingier et al. 2016):

$$\left[(\alpha - 1)\mathcal{F}^2 \frac{h_\infty}{h} + 1 \right] \frac{\partial h}{\partial x} = \tan \theta - \mu(h, U), \quad (4.28)$$

where $\alpha \geq 1$ is the shape factor, h_∞ is the limit of the height at infinity and $\mathcal{F} = U/\sqrt{gh_\infty \cos \theta}$ is the Froude number at infinity. Equation (4.28) was used by Pouliquen 1999a with $\alpha = 1$ (corresponding to a flat velocity profile) and successfully compared to experimental data. In this case, the equation (4.28) reduces to

$$\frac{\partial h}{\partial x} = \tan \theta - \mu(h, U). \quad (4.29)$$

Thus there is no dependence on $\mathcal{F}, h_\infty$ in this case. Saingier et al. 2016 showed that this is a good approximation when the Froude number is small, which is the case in the experiments of Pouliquen 1999a. When $h = 0$, equation (4.29) yields $\partial h / \partial x = \tan \theta - \mu_2$, which means that the front is well defined, with a contact angle equal to $\arctan(\mu_2 - \tan \theta)$.

When the Froude number increases ($\mathcal{F} \geq 1$), experimental investigations of Saingier et al. 2016 showed that the front profile is flattened, and that a good agreement with the theoretical predictions of equation (4.28) is obtained with a shape factor $\alpha = 5/4$, corresponding to a Bagnold profile. In this case, there is still a dependence of the profile on the Froude number, since equation (4.28) can be written

$$\frac{\partial h}{\partial x} = \frac{h[\tan \theta - \mu(h, U)]}{\mathcal{F}^2 h_\infty / 4 + h}. \quad (4.30)$$

A drawback of the choice $\alpha = 5/4$ is that the front is not well defined any more and a precursor film of infinite length appears. Indeed, taking $h = 0$ in the right-hand side of (4.30) leads $\partial h / \partial x = 0$, which means that the slope of the profile goes to zero as h goes to zero, resulting in an exponential tail. Hence the predictions are better with $\alpha = 5/4$, but lead to a nonphysical behaviour near the front.

This paradox might be linked with the fact that the two-equation models with a shape factor different from 1 do not have a good structure (in particular, there is no energy balance equation, see Richard and Gavrilyuk 2012). In Lagrée, Saingier, et al. 2017, the shape factor was computed from two dimensional simulations of dry

fronts based on an approximation of the $\mu(I)$ -rheology (granular RNS/P equations). It was shown that the shape factor is equal to $5/4$ in the flow, except close to the front where it goes to 1. With the three-equation model (4.24), we are able to reproduce this behaviour, since the shape factor can then be seen as an independent variable with its own evolution equation.

In order to study the dry granular fronts given by the regularised model (4.24), we solve it numerically with parameters reproducing the experimental settings. Another possibility would be to write an ODE satisfied by h in the case of a travelling wave solution of (4.24), and to study its properties like it has been done with equation (4.28) for the two-equation model. However, a lot of terms appear in system (4.24), thus the theoretical study is complicated and we choose to solve (4.24) numerically.

The numerical scheme that we use is based on the finite volumes method (Godunov and Bohachevsky 1959; Toro 2013). The system is solved in the conservative variables $(h, hU, h\psi)$. A splitting method is performed in order to treat the conservative part of (4.24) and the source terms separately. The conservative part is calculated with a Rusanov Riemann solver. The source terms are computed explicitly. The hydrostatic pressure gradient is discretized as a source term with central differences, in order to be able to capture the starting and stopping of the flow in other configurations (see section 4). We use a mesh size of 1 mm.

In order to treat numerically the dry fronts, we introduce a height threshold $h_{\min}$. If the value of the height in a cell is under $h_{\min}$, we set the fluxes coming from the cell and the source terms in the cell equal to zero, in order to avoid division by zero. We choose $h_{\min} = 10^{-3}h_{\infty}$ in the case of a dry front, and $h_{\min} = 10^{-3}h_0$ in other cases, where h_0 is the height of the initial condition. In every configuration studied in this work, the height of the flow is less than 1000 particles diameters, which implies that the threshold we choose is less than one particle diameter.

In the case of a dry front, the initial condition is an empty domain, and the boundary condition is given by the equilibrium values $h = h_{\infty}$, $U = 2Ah_{\infty}^{3/2}/5 = U_{\infty}$, $\psi = U_{\infty}^2/(4h^2)$ in the first mesh of the grid. It means that at $t = 0$, a gate of size h_{∞} is opened at the left of the domain, and a flow enters and starts to propagate, with a boundary velocity and enstrophy corresponding to the equilibrium values. After a short transitional regime, the computed solution converges towards a travelling front of height at infinity h_{∞} and of uniform speed U_{∞} (see figure 4.2). The enstrophy is equal to $U_{\infty}^2/(4h^2)$ (i.e. $\alpha = 5/4$), except close to the front where a transition is observed between the regimes where $f = 1$ and where $f = 0$, after which the enstrophy goes to zero (see figure 4.3). The front is well defined, and there is no precursor film.

4. Régularisation et validation du modèle d'écoulement granulaire incompressible –
3. Study of dry fronts

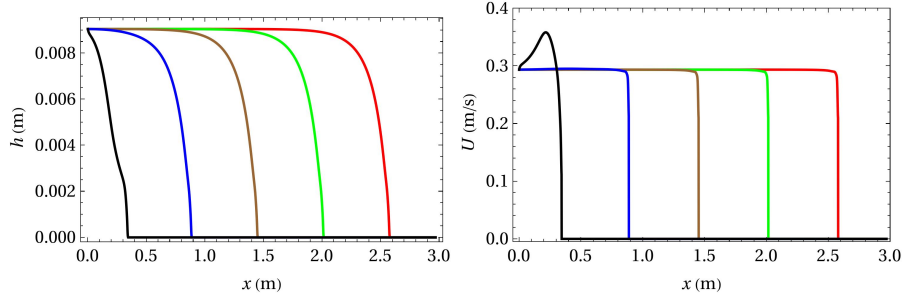


Figure 4.2. – Profiles of height (left) and velocity (right) at different times. Black: $t = 0.71$ s; blue: $t = 2.61$ s; brown: $t = 4.55$ s; green: $t = 6.49$ s, red: $t = 8.43$ s

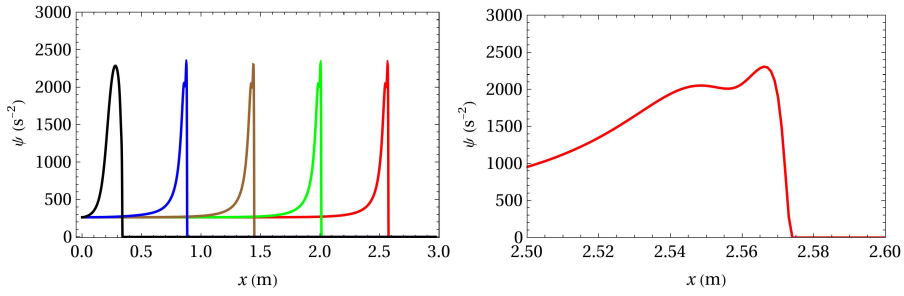


Figure 4.3. – Left: profiles of enstrophy at the same time instants than in figure 4.2. Right: zoom between $x = 2.5$ m and 2.6 m.

The first step is to determine the size of the transition zone near the front between the regimes where $\alpha = 5/4$ and where $\alpha = 1$. This size is controlled by the parameter $\delta > 0$ appearing in the enstrophy equation (4.24c). The smaller δ , the smaller this transition zone. The simulations of Lagrée, Saingier, et al. 2017 show that for the $\mu(I)$ -rheology, this zone is very thin. We find a good agreement between the predictions of our model and the results of Lagrée, Saingier, et al. 2017 for $\delta = 1/20$ (see figure 4.4), thus we keep this value in the rest of the paper. The parameters used for the simulations of figure 4.4 are the same than the ones used by Lagrée, Saingier, et al. 2017: $\mu_1 = 0.35$, $\mu_2 = 0.56$, $I_m = 0.4$, $\phi = 1$, $d = 0.43$ mm. The heights h_∞ and slopes $\tan\theta$ used are $h_\infty = 1.1$ cm for $\tan\theta = 0.37$, $h_\infty = 0.89$ cm for $\tan\theta = 0.4$ and $h_\infty = 0.62$ cm for $\tan\theta = 0.45$, corresponding respectively to a Froude number at infinity of 0.43, 1.03 and 2.09.

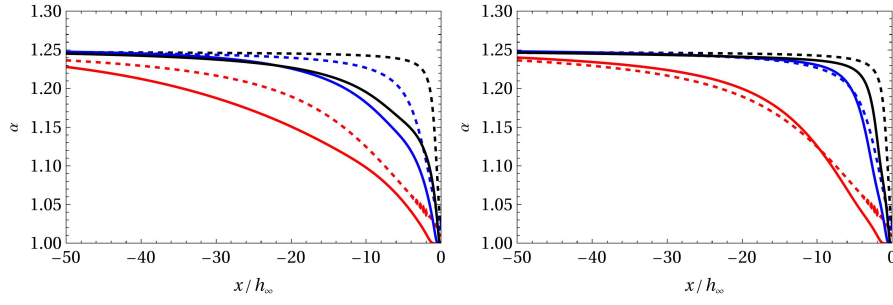


Figure 4.4. – Shape factor of the granular front for three different profiles given by the present model (full line) and the data of Lagrée, Saingier, et al. 2017 (dotted line). Black: $\mathcal{F} = 0.43$; blue: $\mathcal{F} = 1.03$; red: $\mathcal{F} = 2.09$. The value of the parameter δ used in the enstrophy equation (4.24c) is $1/2$ (left) and $1/20$ (right).

We then compare the dry front profiles given by our model with the experimental data of Pouliquen 1999a; Saingier et al. 2016. There is no experimental data of the volume fraction, hence we estimate it with the help of the $\mu(I) - \phi(I)$ -rheology (see Da Cruz et al. 2005):

$$\phi = \phi_c - I_\theta \Delta\phi, \quad \text{with} \quad I_\theta = I_m \frac{\tan \theta - \mu_1}{\mu_2 - \tan \theta} \quad (4.31)$$

the inertial number at equilibrium. We take the standard values $\phi_c = 0.6$ and $\Delta\phi = 0.2$ (Andreotti et al. 2013). The parameters μ_1, μ_2, I_m are determined from the measurements of h_{stop} (see Pouliquen 1999b). In the early work of Pouliquen 1999a, a different form of the friction law was used (exponential form), hence the values of the parameters are different. For comparison with the data of Pouliquen 1999a, we thus use the h_{stop} data of Pouliquen 1999b in order to find the parameters μ_1, μ_2 and I_m corresponding to the friction law we use (fractional form). This was also discussed in the work of Saingier et al. 2016, in which the authors compared different forms of the friction law. For figure 4.5, we use $\mu_1 = \tan(20.65^\circ)$, $\mu_2 = \tan(26.4^\circ)$, $I_m = 0.439$, $d = 1.3$ mm. For figure 4.6, we use $\mu_1 = 0.35$, $\mu_2 = 0.56$, $I_m = 0.2$, $d = 0.5$ mm, as in Saingier et al. 2016. For the comparison with the data of Saingier et al. 2016, we can use the same rheological parameters, since the friction law used by the authors and in our model is the same. The parameters for figure 4.7 are then $\mu_1 = 0.41$, $\mu_2 = 0.76$, $I_m = 0.38$, $d = 0.2$ mm. In some cases, it is difficult to know precisely the position of the front of experimental profiles because there is a gap near the front where no data is available. In these cases, we align the experimental data with the simulated curve by matching the points closest to the front.

4. Régularisation et validation du modèle d'écoulement granulaire incompressible –
3. Study of dry fronts

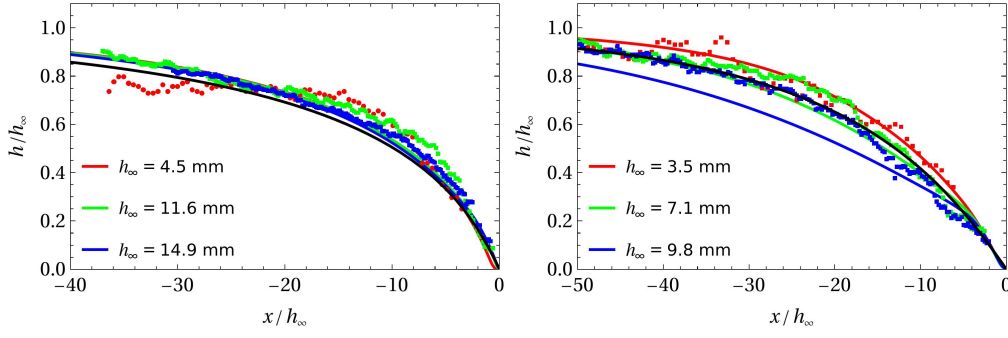


Figure 4.5. – Comparison of the simulated dry fronts from system (4.24) (full lines) with the experimental data of Pouliquen 1999a, system 2 (dots) and the prediction of the two-equation model with $\alpha = 1$ (black line), for $\theta = 22^\circ$ (left) and $\theta = 24^\circ$ (right).

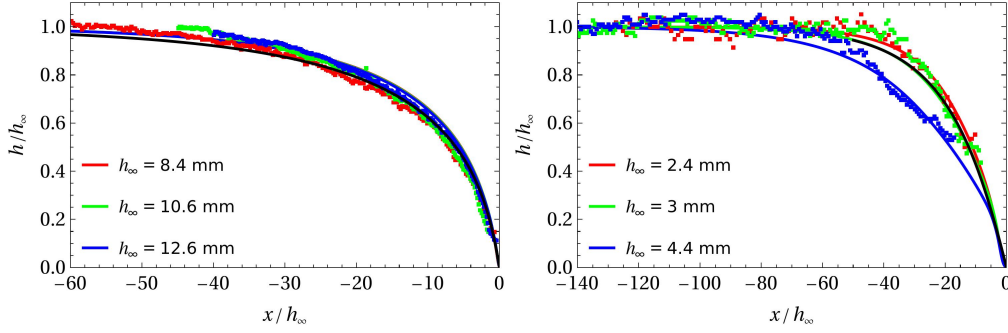


Figure 4.6. – Comparison of the simulated dry fronts from system (4.24) (full lines) with the experimental data of Pouliquen 1999a, system 4 (dots) and the prediction of the two-equation model with $\alpha = 1$ (black line), for $\theta = 21^\circ$ (left) and $\theta = 27^\circ$ (right).

4. Régularisation et validation du modèle d'écoulement granulaire incompressible – 4. Avalanche stopping

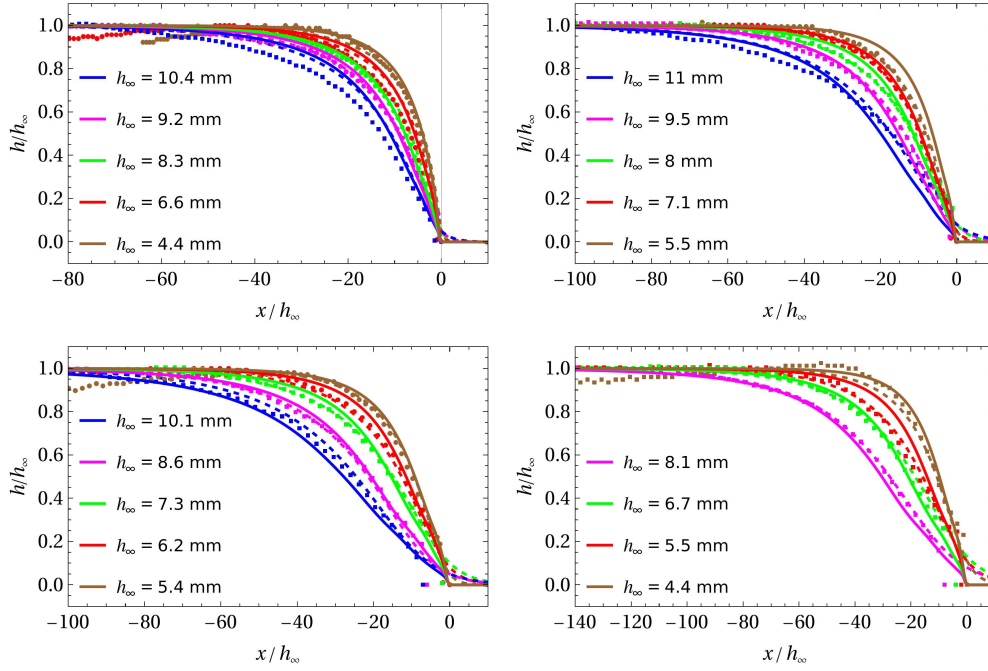


Figure 4.7. – Comparison of the simulated dry fronts from system (4.24) (full lines) with the experimental data of Saingier et al. 2016 (dots) and the predictions of the two-equation model with $\alpha = 5/4$ (dotted lines). The values of θ are 26.2° (top left), 27.2° (top right), 28.2° (bottom left) and 29.2° (bottom right).

Overall, we find a good agreement between the predictions of the three-equation system (4.24), the predictions of the two-equation model with a shape factor $\alpha = 5/4$ (or $\alpha = 1$ when the Froude number is small), and experimental data in the whole range of investigated parameters. The worst prediction is the one of the blue curve corresponding to $h_\infty = 9.8$ mm and $\theta = 24^\circ$ in figure 4.5. For this curve, the Froude number is not small ($\mathcal{F} \approx 1.2$) and the prediction of the three-equation model agrees with the one the two-equation model with shape factor $\alpha = 5/4$, which also underestimates the height of the profile. Apart from this curve, we obtain a good accuracy with experimental data. Note that the three-equation model is able to capture the flattening of the profile with the increasing of the Froude number and yields a well defined front at the same time, which is not the case with a two-equation model. This fact motivates the use of system (4.24) for the modelling of dry fronts.

4. Avalanche stopping

Another successful prediction of the two-equation model with friction law (and shape factor $\alpha = 1$) is the stopping of a granular flow when the inclination of the bottom is smaller than the static angle of repose (see for instance Pouliquen and Forterre 2002; Mangeney et al. 2007; H. A. Martin et al. 2023; Mangeney-Castelnau,

Bouchut, et al. 2005; Rocha et al. 2019; Poulain et al. 2023). A complex feature of this configuration is the transition between static and flowing regimes. This transition is enabled by the fact that when the medium is at rest, friction exactly balances the other forces (weight and hydrostatic pressure gradient) acting on the medium, up to a maximal threshold beyond which the flow starts.

From a numerical point of view, it is not straightforward to capture this transition. The notion of well-balanced scheme was introduced by Gosse et al. 1996; Greenberg et al. 1996 to designate the schemes that preserve steady solutions. The hydrostatic reconstruction method of Audusse et al. 2004 provides a well balanced scheme for the shallow water equations with topography. It was used to write the code SHALTOP, which is a well-balanced scheme for the shallow water equations with granular friction and arbitrary topography (Bouchut, Mangeney-Castelnau, et al. 2003; Bouchut and Westdickenberg 2004; Mangeney-Castelnau, Bouchut, et al. 2005; Mangeney et al. 2007). In this code, the pressure gradient, the weight and the friction are discretized as flux terms, in such a way that the scheme preserves solutions at rest. For an introduction to well-balanced schemes, we refer to Bouchut 2004. In this work, we choose to follow the method proposed by Denisenko 2024; Denisenko, Richard, et al. 2025: we thus discretize the pressure gradient, the weight and the friction as source terms and apply a splitting method between the conservative part (without hydrostatic pressure) and the source terms (including hydrostatic pressure). This way, we obtain a scheme that preserves solutions at rest, which are for system (4.24) solutions such that $U = 0$, $\psi = 0$ and the friction balances the weight and pressure gradient.

To evaluate the performances of system (4.24) in the regime of small inclination angles θ , we simulate a granular dam break and compare the results with the experimental data of Farin et al. 2014. Note that we are in the case in which the slope $\tan \theta$ is smaller than μ_1 , hence $f = 0$ and system (4.24) reduces to the simpler version

$$\left\{ \begin{array}{l} \frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0, \end{array} \right. \quad (4.32a)$$

$$\left\{ \begin{array}{l} \frac{\partial hU}{\partial t} + \frac{\partial}{\partial x} \left(hU^2 + \beta h^3 \psi + K \frac{gh^2}{2} \cos \theta \right) = gh \cos \theta [\tan \theta - \mu(h, U)], \end{array} \right. \quad (4.32b)$$

$$\left\{ \begin{array}{l} \frac{\partial h\psi}{\partial t} + \frac{\partial hU\psi}{\partial x} = \frac{\alpha_3}{h^{1/2}} \left(\psi - e^{-\frac{5d}{h}} \frac{U^2}{4h^2} \right). \end{array} \right. \quad (4.32c)$$

We use for the simulations the parameters that were taken in H. A. Martin et al. 2023: $\mu_1 = 0.48$, $\mu_2 = 0.73$, $I_0 = 0.279$, $d = 0.7$ mm, $\phi = 1$. The initial condition is a rectangle of granular media of length R_0 and height aR_0 , where a is the aspect ratio. We investigate the aspect ratios $a = 0.7$, $a = 0.31$ and $a = 1.24$, with respective lengths $R_0 = 20$ cm, $R_0 = 30$ cm and $R_0 = 15$ cm, for two inclination angles $\theta = 0^\circ$ and $\theta = 16^\circ$. The initial volume is delimited by a wall on the left, the inclined plane below, the free surface above and a gate on the right, which is opened at $t = 0$. For the wall boundary condition, we add a phantom cell, in which h and ψ are equal to the height and enstrophy of the first cell, and U is opposed to the velocity of the first cell. The results are shown in figure 4.8.

4. Régularisation et validation du modèle d'écoulement granulaire incompressible – 4. Avalanche stopping

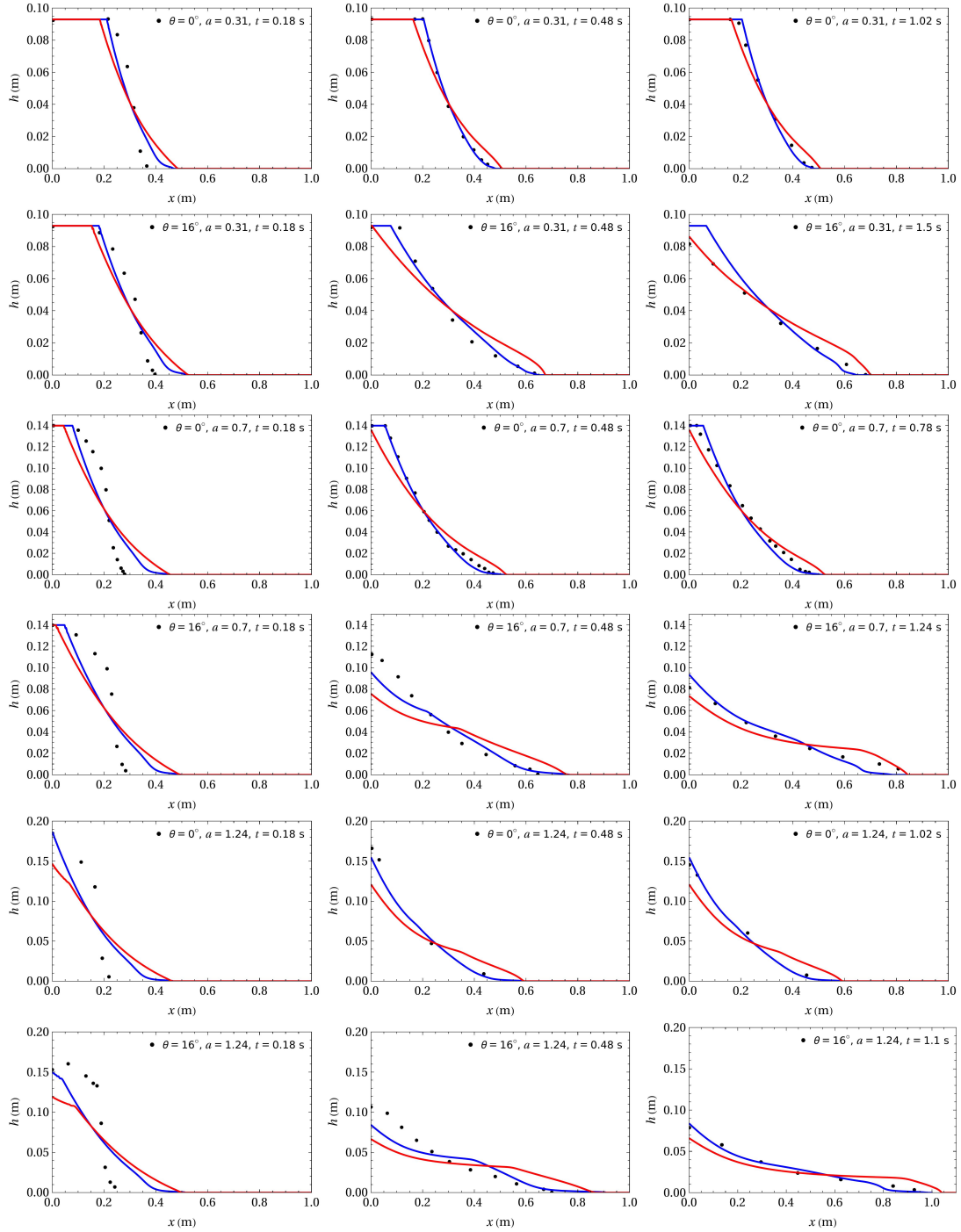


Figure 4.8. – Profiles of height after the granular dam break. Black dots: experimental data of Farin et al. 2014, extracted from the figures of H. A. Martin et al. 2023. In blue, three-equation model (4.24). In red, two-equation model.

The results are in good agreement with the experimental data. As mentioned in H. A. Martin et al. 2023, the two-equation model overestimates the initial spreading, and gives an accurate final deposit. This is also true for the three-equation model used

here. We note that the overestimation of the initial spreading is reduced by the use of the three-equation model. For the two models, this overestimation increases with the aspect ratio. Also, the prediction of the final deposit is generally better, except in the case $\theta = 16^\circ$, $a = 0.31$. The position of the front is very similar for the two models, with a difference inferior to 5 cm in all the pictures, corresponding to a typical relative deviation between 0% and 12.5%.

5. Accelerated flows

The last case we study is a simplified imitation of a scenario that can lead to an avalanche, including both an acceleration zone where $\tan\theta > \mu_2$ and a stopping zone. To study this configuration, we reproduce numerically the experiment of Poulain et al. 2023. In their setup, they use a channel divided into two portions. The first portion is a slope of variable inclination angle. The angles used in the experiment are $\theta = 35^\circ$ and $\theta = 45^\circ$. The bottom of the slope is connected to the second portion of the channel which is a horizontal bed. The granular mass is initially at rest on the slope thanks to a vertical gate. The opening of the gate at $t = 0$ makes the flow spread on the slope, until it reaches the horizontal bed and stops.

We use the parameters that were used in Poulain et al. 2023: $\mu_1 = 0.406$, $\mu_2 = 0.620$, $\mu_3 = 0.430$, $d = 4$ mm, $L = 1.3d$, $\beta_* = 0.136$. The friction law used in Poulain et al. 2023 when $\mathcal{F} > \beta_*$ reads

$$\mu(h, U) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + \frac{h^{3/2}\beta_*\sqrt{g\cos\theta}}{LU}}. \quad (4.33)$$

In order to have the same expression, it means that with the notations we use we must choose I_m and ϕ such that $I_m\sqrt{\phi} = 5\beta_*/2.6$ (see (4.7) and (4.19)). We choose $\phi = 0.6$, which yields $I_m = 5\beta_*/(2.6\sqrt{\phi}) = 0.338$. Note that ϕ is not constant throughout the experiment, and the authors of Poulain et al. 2023 estimated that the maximum dilatation is 27% when $\theta = 35^\circ$, and 35% when $\theta = 45^\circ$. However, system (4.24) is derived from the incompressible $\mu(I)$ -rheology, for which ϕ is assumed to be constant. As a consequence, our model does not take dilatancy into account. As in Poulain et al. 2023, the transition between the slope and the horizontal bed is smoothed with a second order polynomial for the numerical simulations. Note that topography variations can cause curvature effects, that are not taken into account in our model. The results are shown in figures 4.9 and 4.10.

4. Régularisation et validation du modèle d'écoulement granulaire incompressible –
5. Accelerated flows

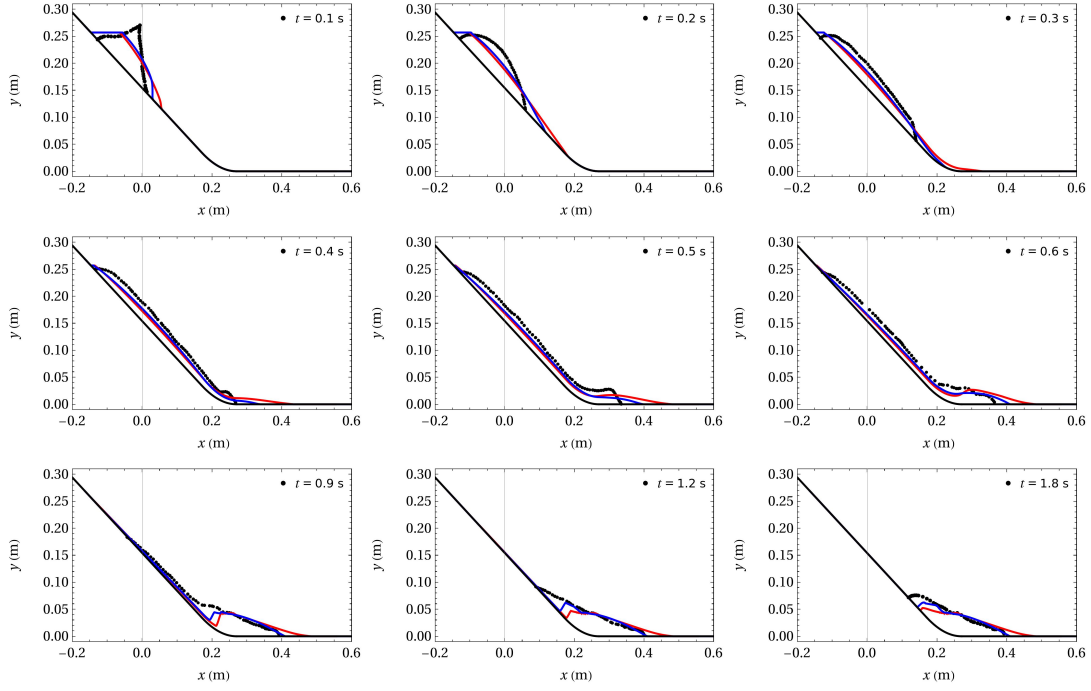


Figure 4.9. – Profiles of height after the granular dam break on the 35° slope. Black dots: experimental data of Poulain et al. 2023. In blue, three-equation model (4.24). In red, two-equation model.

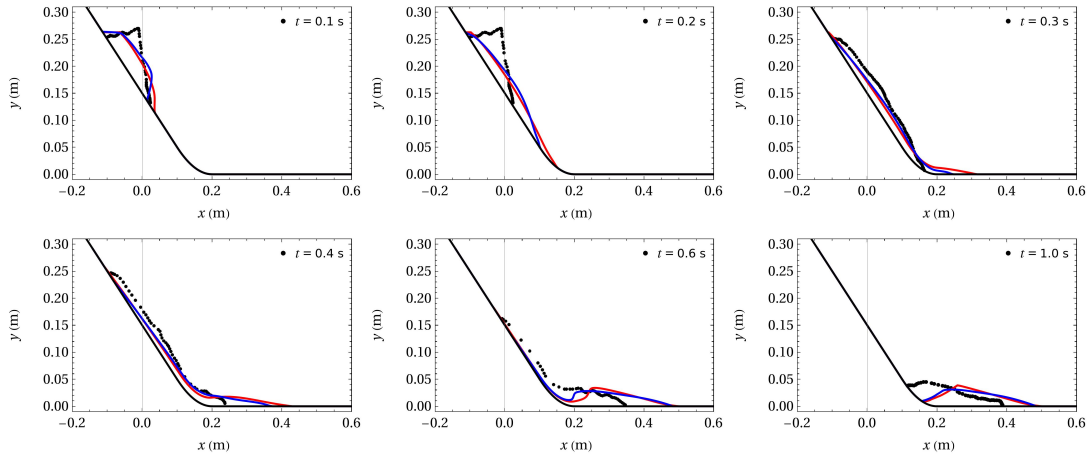


Figure 4.10. – Profiles of height after the granular dam break on the 45° slope. Black dots: experimental data of Poulain et al. 2023. In blue, three-equation model (4.24). In red, two-equation model.

We observe that the flow starts too fast, as in section 4. As a consequence, the height profiles of the numerical solutions are thinner and more spread than the ones of the experimental data. This discrepancy is amplified by the dilatation of the flow, which was measured in the experiments but is not reproduced in the numerical simulations.

As a consequence, the area under the black dots increases during the accelerating phase, while the areas under the red and blue curves stay constant throughout all the simulation (this is a consequence of the incompressibility assumption). Again, the initial motion is better predicted with the three-equation model. The position of the final deposit is well predicted for $\theta = 35^\circ$, and is 10 cm too far for $\theta = 45^\circ$. These results suggest that the regularised three-equation model stays pertinent even in this more realistic situation incorporating a wide range of slopes and which is really far from equilibrium configurations.

6. Conclusion

In this work, we present a regularised three-equation model consistent with the $\mu(I)$ -rheology. The model is derived from a previous version which is only valid close to stationary and uniform flows (see Deleage and Richard 2025). To obtain the regularised version, a parameter f is introduced, that enables to obtain a well-posed model for any slope. The parameter f is carefully written in such a way that the consistency of the model is not affected by the regularisation procedure. Furthermore, the linearised model around a stationary uniform flow stays unchanged upon the regularisation process, which means that the linear properties studied in Deleage and Richard 2025 are also valid for the regularised model.

The accuracy of the model is also investigated in different flow regimes that are far from stationary uniform flows. We first study the case of travelling dry fronts. We show that the model is able to capture the flattening of the front when the Froude number increases and to have a well defined front at the same time. This is a significant improvement of the three-equation model, since these two properties cannot be obtained simultaneously with a two-equation model. Furthermore, a good agreement is obtained with experimental data in the large range of investigated parameters. We then study the spread of a granular mass after a dam break on small slopes. We again obtain a good accuracy, and we show that the three-equation model is better close to the start of the flow. We also observe generally a better prediction of the final deposit obtained when the flow stops. Finally, we study accelerated flows on high slopes. For this last test, we also consider topography variations. We show that the predictions of the model stay valid even in this strongly unsteady configuration.

A future development of this work could be the writing of a second order version of the model presented here, including diffusive terms. These terms could be written with the help of a regularised version of the diffusion coefficient obtained in Gray and Edwards 2014; Deleage and Richard 2025. A further development of this work would be to take dilatancy into account by considering a compressible rheology. Indeed, these effects are important for the modelling of flows on high inclination slopes.

5. Un modèle moyenné sur la profondeur pour les écoulements granulaires consistant avec la rhéologie $\mu(I)$ compressible

Dans ce chapitre, on dérive un modèle moyenné sur la profondeur consistant avec une rhéologie granulaire compressible. La consistance à l'ordre 1 est obtenue grâce à un développement asymptotique en puissances du paramètre de couche mince jusqu'à l'ordre 1. Les équations du modèle sont écrites à l'aide de moyennes de Favre. On obtient un modèle à trois équations d'évolution, compatibles avec un bilan d'énergie. Les variables sont le "hold-up" (correspondant à l'intégrale de la fraction volumique sur la profondeur), la vitesse moyenne et l'ensrophie. On donne également des formules permettant de reconstruire la hauteur, la fraction volumique moyenne et le profil de vitesse à partir de ces trois variables. On montre que, par rapport au modèle obtenu au chapitre 3, la compressibilité peut être prise en compte en ajoutant un facteur correctif dans la loi de friction, et en redéfinissant une des constantes du modèle. On étudie ensuite l'instabilité des ondes de surface. En particulier, on obtient pour la première fois une expression pour le seuil d'instabilité théorique de la rhéologie $\mu(I) - \phi(I)$ dans la limite des ondes longues. En effet, ce seuil est le même pour la rhéologie et pour le modèle moyenné grâce à la consistance jusqu'à l'ordre 1. On propose également une version du modèle avec des termes additionnels de diffusion permettant de retrouver une fréquence de coupure. La méthode utilisée pour dériver le modèle est identique à celle présentée au chapitre 3. Par conséquent, la structure de ce chapitre est similaire à celle du chapitre 3.

1. Introduction

The study of granular flows is an active field of research, motivated by the modelling of natural hazards, such as avalanches, and of industrial processes. Constituted by a large number of solid particles, granular media exhibit a rich diversity of behaviours that makes them complex to describe in a unified way, such as hysteresis, liquid/solid transitions, liquid/gas transitions (see Andreotti et al. 2013).

Granular fluids can be obtained on inclined bottoms. The motion is then driven

5. Modèle consistant d'écoulement granulaire compressible – 1. Introduction

by a competition between gravity acceleration and dissipative interactions (friction and collisions), occurring between the particles themselves, and between the particles and the bottom. In this configuration, depth-averaged models can be used in order to describe the flow. The benefits of these models is that they are well-posed and fast to solve numerically. A depth-averaged model describing the flow of a granular mass on a topography was proposed by Savage et al. 1989. The model is a two-equation model, giving the evolution of the height h and of the depth-averaged horizontal velocity U of the flow. The granular properties of the medium were taken into account with the help of a Coulomb friction law. This model was generalised to two dimensions and to variable topographies, and used to describe natural hazards (see for instance Denlinger et al. 2001; Mangeney-Castelnau, Bouchut, et al. 2005; Denlinger et al. 2004; Larrieu et al. 2006; Peruzzetto et al. 2019).

An important modification to the model of Savage et al. 1989 was proposed by Pouliquen 1999b. Instead of considering a constant friction coefficient, Pouliquen 1999b proposed a variable coefficient $\mu(h, U)$, depending on the height and the velocity of the flow. This hypothesis enabled to match the observations of the existence of steady uniform granular flows for different values of the slope (Pouliquen 1999b). A non-monotonic version of the friction law of Pouliquen 1999b was also proposed by Pouliquen and Forterre 2002, in order to capture better the flow stopping. This model was applied to study the spreading of a granular mass (e.g. Pouliquen and Forterre 2002; Mangeney-Castelnau, Vilotte, et al. 2003; Mangeney et al. 2007).

The idea of a variable friction coefficient was then used to write a local rheology, first in simple shear configurations (GdR MiDi 2004), then in the general three dimensional case (Jop et al. 2006), by using the assumption of a constant solid volume fraction ϕ . The dependence on the friction coefficient μ on the variables h and U was replaced by a dependence on a local dimensionless parameter I , called inertial number. This number can be interpreted as the ratio between a microscopic timescale and a macroscopic timescale (GdR MiDi 2004). The rheology was thus called $\mu(I)$ -rheology, or incompressible $\mu(I)$ -rheology. For numerical simulations of granular flows with the $\mu(I)$ -rheology, see for instance Lagrée, Staron, et al. 2011; N. Martin et al. 2017; H. A. Martin et al. 2023.

The $\mu(I)$ -rheology was also used to improve the depth-averaged model of Pouliquen and Forterre 2002. A depth-averaged diffusion term was added to the model by Forterre 2006 and Gray and Edwards 2014. This term enabled to recover the cut-off frequency of the roll waves instability that was observed in the experiments of Forterre and Pouliquen 2003. For other applications of the model of Gray and Edwards 2014, see for instance Edwards and Gray 2015; Edwards, Viroulet, et al. 2017; Kanellopoulos 2025. The Bagnold velocity profile which is predicted by the $\mu(I)$ -rheology (see e.g. GdR MiDi 2004) was also taken into account by adding to the model a shape factor of $5/4$ in the depth-averaged momentum flux. Saingier et al. 2016 showed that it enables a better prediction of granular fronts. However, they also showed that a nonphysical precursor film appeared due to the shape factor of $5/4$. Furthermore, the predictions of the model with a shape factor of $5/4$ are not in good agreement with the experimental data of Forterre and Pouliquen 2003 concerning the roll waves

5. Modèle consistant d'écoulement granulaire compressible – 1. Introduction

instability.

This paradox was solved by Deleage and Richard 2025, who derived a three-equation model with an additional variable called enstrophy, enabling to take into account vertical variations of the velocity. The derivation used a method proposed by Teshukov 2007 and applied to various contexts (see for instance Richard and Gavriluk 2012; Richard, Ruyer-Quil, et al. 2016; Denisenko, Richard, et al. 2023). The derivation was made assuming an asymptotic expansion of the variables in powers of the shallow water parameter, in order to obtain consistency at order 1 with the rheology and an exact stability threshold for the roll waves instability. The asymptotic expansions were also used to derive depth-averaged models by e.g. Ruyer-Quil et al. 2000; Noble et al. 2013 in other situations. A second-order version of the model with diffusion similar to the one found by Gray and Edwards 2014 was also proposed by Deleage and Richard 2025.

A limit of the $\mu(I)$ -rheology is that it becomes ill-posed at low and high inertial number, as shown by Barker, Schaeffer, Bohórquez, et al. 2015. Indeed, in these regimes, it admits Hadamard instability, which means that the linear growth rate of perturbations diverges in the high wave number limit. As a consequence, numerical computations are grid dependent when the mesh size goes to zero (see for instance N. Martin et al. 2017).

Another defect of the rheology is the hypothesis that the volume fraction ϕ is a constant. Indeed, Da Cruz et al. 2005 showed with discrete simulations that the volume fraction also varies with the inertial number. They thus proposed a scalar rheology, called $\mu(I) - \phi(I)$ rheology, which is valid in simple shear configurations. The $\mu(I) - \phi(I)$ rheology is thus a generalisation of the local rheology proposed by the GdR MiDi 2004. Several attempts were made to generalise the $\mu(I) - \phi(I)$ rheology into a tensorial constitutive law, as it was done by Jop et al. 2006 in the incompressible case. A difficulty is that, in the compressible case, the pressure cannot be defined with the help of the incompressibility condition. Thus another relation linking the pressure and the flow variables must be proposed.

A first approach was used by Nott 2009; Trulsson et al. 2013; Heyman et al. 2017. They used the definition of the inertial number in order to define an equilibrium pressure, depending on the volume fraction and on the shear rate of the flow. The equilibrium pressure then enabled to define the deviatoric stress tensor, with the help of two friction coefficients: the usual friction coefficient of the incompressible $\mu(I)$ -rheology, associated to shear deformations, and an additional bulk friction coefficient, associated to volume changes. Heyman et al. 2017 showed that this rheology enabled to obtain well-posed equations (i.e. without Hadamard instability) when the value of the bulk friction is high enough.

Another approach was used by Barker, Schaeffer, Shearer, et al. 2017 (see also Andrade et al. 2012). Their idea is to write a dilatancy law linking the pressure, the divergence of the velocity, the inertial number and the solid volume fraction. This relation is written in such a way that one recovers the equality $\phi = \phi(I)$ when the flow is incompressible. The model was inspired from critical state soil mechanics (Schofield et al. 1968; Jackson 1983) and was called compressible I -dependent rheology (CIDR).

Barker, Schaeffer, Shearer, et al. 2017 showed that, under some assumptions on the dilatancy law used and on the norm of the deviatoric stress tensor, the CIDR constitutive equations are well-posed, i.e. they do not exhibit Hadamard instability. The CIDR equations as well as the constitutive equations proposed by Heyman et al. 2017 were commented by Goddard et al. 2018.

At the depth-averaged level, a lot of work has been made to introduce dilatancy effect for immersed granular media, also called suspensions (see for instance the works of Pailha et al. 2009; Iverson et al. 2014; Bouchut, Fernández-Nieto, et al. 2016; Meng et al. 2018; Sun et al. 2024; Bouchut, Drach, et al. 2025). In the case of a dry granular flow, Bouchut, Fernández-Nieto, et al. 2021 proposed a three-equation model for the variables ϕ , h and U . The model is derived from a compressible rheology, for which the dilatancy is taken into account via a dilatancy angle, as introduced by Roux et al. 1998. The model of Bouchut, Fernández-Nieto, et al. 2021 was used to simulate the collapse of a granular column.

In this work, we aim to extend the previous work of Deleage and Richard 2025 to the case of the compressible $\mu(I) - \phi(I)$ rheology. We thus aim to derive a three-equation model for granular flow, that is able to take into account shear effects and dilatancy. Another objective is to write a model which is fully consistent up to order 1 with the rheology, in order to obtain the good stability threshold for the roll waves instability. In section 2, we present the rheology that is used and the non dimensionalisation of the equations. In section 3, we perform an asymptotic expansion of the variables up to order 1. In section 4, we show the depth-averaging procedure. The resulting model and its main properties are studied in section 5.

2. Governing equations

2.1. Constitutive equations

We study the flow of a granular fluid, which is described by its density ρ and its velocity $\mathbf{v}$. We assume that the fluid is made of solid particles. Hence the density is proportional to ϕ the volume fraction of the beads and is written

$$\rho = \rho_p \phi, \quad (5.1)$$

where the constant ρ_p is the density of the beads. We assume that the fluid is flowing down an inclined plane that makes an angle θ with the horizontal. The motion of the fluid is described by the mass and momentum equations, which read

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) &= 0, \\ \frac{\partial \rho \mathbf{v}}{\partial t} + \operatorname{div}(\rho \mathbf{v} \otimes \mathbf{v}) &= \operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{g}. \end{aligned} \quad (5.2)$$

5. Modèle consistant d'écoulement granulaire compressible – 2. Governing equations

The second order tensor $\boldsymbol{\sigma}$ is called the Cauchy stress tensor and can be written

$$\boldsymbol{\sigma} = -p\mathbf{I}_d + \boldsymbol{\tau}, \quad (5.3)$$

where $\mathbf{I}_d$ is the identity tensor. The quantity p is the pressure and $\boldsymbol{\tau}$ is the deviatoric stress tensor. To express the deviatoric stress tensor, we first split the strain-rate tensor

$$\dot{\boldsymbol{\gamma}} = \mathbf{grad} \, \boldsymbol{v} + (\mathbf{grad} \, \boldsymbol{v})^T \quad (5.4)$$

into a deviatoric part and an isotropic part:

$$\dot{\boldsymbol{\gamma}} = \mathbf{S} + \text{div}(\boldsymbol{v})\mathbf{I}_d, \quad \text{i.e.} \quad \mathbf{S} = \dot{\boldsymbol{\gamma}} - \text{div}(\boldsymbol{v})\mathbf{I}_d \quad (5.5)$$

such that $\text{Tr}(\mathbf{S}) = 0$. The deviatoric stress tensor is then linked to the deviatoric strain-rate tensor via an effective viscosity η_e :

$$\boldsymbol{\tau} = \eta_e \mathbf{S}. \quad (5.6)$$

In the isotropic case, η_e is a scalar. In order to define the effective viscosity, we introduce the inertial number I , which is a dimensionless parameter given by the relation

$$I = \frac{|\mathbf{S}|d}{\sqrt{p/\rho_p}}. \quad (5.7)$$

The quantity d is the diameter of a bead, and $|\mathbf{S}|$ is the norm of the second order tensor $\mathbf{S}$, defined by

$$|\mathbf{S}| = \sqrt{\frac{\mathbf{S} : \mathbf{S}}{2}}. \quad (5.8)$$

Note that, in the incompressible case, the inertial number is defined with the norm of the strain-rate tensor $|\dot{\boldsymbol{\gamma}}|$ (GdR MiDi 2004). In the compressible case, it has to be replaced by the norm of $\mathbf{S}$, as shown by the discrete simulations of Cortet et al. 2009; Lacaze et al. 2009. We also introduce the friction coefficient $\mu(I)$, which depends only on I . The dependence is experimentally determined with three constants μ_1 , μ_2 and I_m as (Jop et al. 2006)

$$\mu(I) = \mu_1 + \frac{\mu_2 - \mu_1}{1 + I_m/I}. \quad (5.9)$$

The two constants μ_1 and μ_2 are limit cases: the value μ_2 is the maximal value of $\mu(I)$, obtained when $I \rightarrow \infty$, the value μ_1 is the minimal value, obtained when $I = 0$, and the granular fluid only flows when $|\boldsymbol{\tau}| > \mu_1 p$. Note that this relation can be inverted to write

$$I = I_m \frac{\mu - \mu_1}{\mu_2 - \mu}. \quad (5.10)$$

The expression of the effective viscosity is then

$$\eta_e = \mu(I) \frac{p}{|\mathbf{S}|}. \quad (5.11)$$

The equations are complemented by a relation linking the volume fraction ϕ with the inertial number I : $\phi = \phi(I)$. The volume fraction is a decreasing function of the inertial number (dilatancy). A typical relation is an affine law (Andreotti et al. 2013; Da Cruz et al. 2005)

$$\phi(I) = \phi_c - (\phi_c - \phi_m)I, \quad (5.12)$$

where $\phi_c > \phi_m$ are two constants (ϕ_c can be interpreted as the maximal volume fraction). This law has been validated with numerical simulations when $I < 1$ (Da Cruz et al. 2005 in the case of a plane shear, see also for instance Faug, Beguin, et al. 2009, figure 2.b and Rincón 2024, figure 3.15b for the case of a free surface flow on an inclined plane) but gives a negative volume fraction for greater values of I , which is not admissible. Deviations to this affine law were observed when $I \geq 1$ by Faug, Beguin, et al. 2009 (figure 2.b). In the following, we will only assume that ϕ is a given decreasing and positive function of I . As a consequence, the relation $\phi = \phi(I)$ can be inverted to obtain the inertial number from the volume fraction: $I = I(\phi)$. Note that the stress tensor can be expressed with the only help of the two variables ϕ and $\mathbf{v}$:

$$\boldsymbol{\sigma} = -p(\phi, |\mathbf{S}|)\mathbf{I}_d + \mu[I(\phi)]p(\phi, |\mathbf{S}|)\frac{\mathbf{S}}{|\mathbf{S}|}, \quad \text{with} \quad p(\phi, |\mathbf{S}|) = \rho_p \frac{d^2 |\mathbf{S}|^2}{I(\phi)^2}. \quad (5.13)$$

A slightly more general version was considered by Nott 2009; Trulsson et al. 2013; Heyman et al. 2017, with a bulk friction coefficient $\mu_v(\phi)$, associated to volume changes:

$$\boldsymbol{\sigma} = -p(\phi, |\mathbf{S}|)\mathbf{I}_d + \mu[I(\phi)]p(\phi, |\mathbf{S}|)\frac{\mathbf{S}}{|\mathbf{S}|} + 2\mu_v(\phi)p(\phi, |\mathbf{S}|)\frac{\text{div}(\mathbf{v})}{|\mathbf{S}|}\mathbf{I}_d. \quad (5.14)$$

The advantage of considering an additional bulk friction coefficient is that the equations become well-posed (i.e. there is no Hadamard instability) as soon as

$$\mu_v(\phi) > 1 - 7\mu[I(\phi)]/6 \quad (5.15)$$

(Heyman et al. 2017). Furthermore, the energy dissipation is always negative (i.e. $\boldsymbol{\sigma} : \dot{\boldsymbol{\gamma}} < 0$, for any value of $|\mathbf{S}|$ and $\text{div}(\mathbf{v})$), if and only if $\mu_v(\phi)\mu[I(\phi)] > 1/8$, as noted by Goddard et al. 2018. However, if the bulk friction coefficient μ_v is used, it is an additional parameter of the model. Hence a precise value is needed for applications. This value could be estimated empirically through experimental measurements, as it was done for the $\mu(I)$ curve. To the best of our knowledge, the precise value of this coefficient is not known. Furthermore, if one uses a bulk friction coefficient different from zero, the pressure cannot be defined with the only variables ϕ and $|\mathbf{S}|$, and a dependence on $\text{div}(\mathbf{v})$ has to be added (Heyman et al. 2017). Thus in the definition of the inertial number (5.7), the pressure p has to be replaced by the expression $p(\phi, |\mathbf{S}|)$ (5.13), which was called the equilibrium pressure by Heyman et al. 2017. For these reasons, we choose to present here the derivation in the case $\mu_v = 0$. However, as it is shown in appendix 7.1, the addition of a bulk friction coefficient in the constitutive equations only results in adding terms that are of order $O(\varepsilon^2)$ in the shallow water asymptotic, as soon as the bulk friction coefficient μ_v is at most of order $O(1)$ with

respect to ε . Thus the derivation presented here and the model obtained are consistent with the general rheology (5.14), whatever the value of the bulk friction coefficient.

We denote by z the coordinate in the direction normal to the inclined plane and by x the coordinate in the direction parallel to the inclined plane (see Figure 5.1). The components of the velocity $\mathbf{v}$ are u and w in the x -direction and in the z -direction respectively.

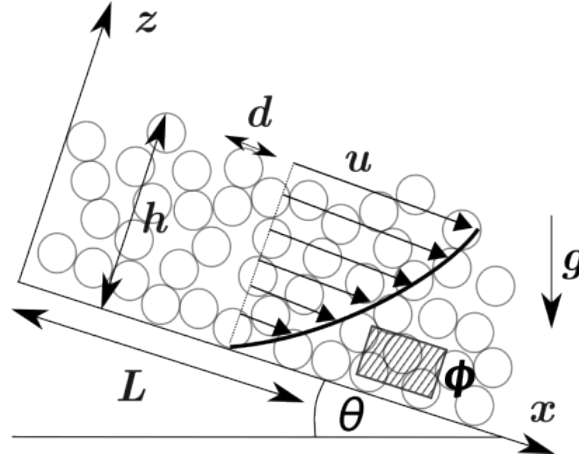


Figure 5.1. – Definition sketch.

The system (5.2) admits an energy balance equation. The mechanical energy density E of the fluid is equal to the sum of the kinetic energy density and the gravitational potential energy density:

$$E = \frac{1}{2} \rho |\mathbf{v}|^2 - \rho g x \sin \theta + \rho g z \cos \theta. \quad (5.16)$$

The energy balance equation is then

$$\frac{\partial E}{\partial t} + \operatorname{div}(E\mathbf{v}) = \mathbf{div}(\boldsymbol{\sigma}) \cdot \mathbf{v}. \quad (5.17)$$

The granular fluid is assumed to be delimited downwards by the inclined plane at $z = 0$, and a free surface upwards at $z = h$. The field h is a function of (t, x) . For any field f depending on the variable z , we write $f(0) := f|_{z=0}$, and $f(h) := f|_{z=h}$. We denote by $\mathbf{n}(t, x)$ the unit vector normal to the free surface and pointing outward the flow. It is given by

$$\mathbf{n} = \frac{1}{\sqrt{1 + |\partial h / \partial x|^2}} \left(-\frac{\partial h}{\partial x}, 1 \right)^T. \quad (5.18)$$

We can now define the boundary conditions satisfied by the velocity:

- At $z = 0$, we assume a non-penetration condition given by

$$w(0) = 0, \quad (5.19)$$

as well as a no-slip condition, given by

$$u(0) = 0. \quad (5.20)$$

- At $z = h$, the velocity drives the evolution of the free surface via the kinematic boundary condition

$$\frac{\partial h}{\partial t} + u(h) \frac{\partial h}{\partial x} = w(h). \quad (5.21)$$

The dynamic boundary condition states that the normal stresses are continuous across the free surface. The normal stress above the free surface is given by the atmospheric pressure p_{atm} . Without loss of generality, this constant can be taken equal to zero. This leads to the equation

$$\boldsymbol{\sigma}(h) \cdot \mathbf{n} = 0, \quad (5.22)$$

which expands as

$$\left\{ \begin{array}{l} \tau_{xz}(h) + p(h) \frac{\partial h}{\partial x} - \tau_{xx}(h) \frac{\partial h}{\partial x} = 0, \\ \tau_{zz}(h) - p(h) - \tau_{xz}(h) \frac{\partial h}{\partial x} = 0. \end{array} \right. \quad (5.23a)$$

$$\left\{ \begin{array}{l} \tau_{xz}(h) + p(h) \frac{\partial h}{\partial x} - \tau_{xx}(h) \frac{\partial h}{\partial x} = 0, \\ \tau_{zz}(h) - p(h) - \tau_{xz}(h) \frac{\partial h}{\partial x} = 0. \end{array} \right. \quad (5.23b)$$

2.2. Dimensionless form of the equations

We write here all equations in a non-dimensional form. In order to do so, we denote, for any field f , a corresponding dimensionless field by $\tilde{f}$.

We assume here that the flow occurs at a typical horizontal scale L_0 , and a typical vertical scale h_0 , such that $\varepsilon = h_0/L_0 \ll 1$ is a small parameter. We also assume that the horizontal velocity is of order u_0 , such that the Froude number

$$F = \frac{u_0}{\sqrt{gh_0}} \quad (5.24)$$

is of $O(1)$, as it is the case for instance in the experiments performed by Pouliquen [1999b](#) and Forterre and Pouliquen [2003](#). We introduce the following dimensionless variables:

$$\tilde{x} = \frac{x}{L_0}, \quad \tilde{z} = \frac{z}{h_0}, \quad \tilde{h} = \frac{h}{h_0}. \quad (5.25)$$

The time and velocity are scaled according to the usual shallow-water scaling as

$$\tilde{t} = t \frac{u_0}{L_0}, \quad \tilde{u} = \frac{u}{u_0}, \quad \tilde{w} = \frac{w}{\varepsilon u_0}, \quad (5.26)$$

The dimensionless pressure, density and size of the beads are found by

$$\tilde{p} = \frac{p}{\rho_p g h_0}, \quad \tilde{\rho} = \frac{\rho}{\rho_p} = \phi, \quad \tilde{d} = \frac{d}{h_0}. \quad (5.27)$$

5. *Modèle consistant d'écoulement granulaire compressible* – 2. *Governing equations*

We also scale the deviatoric stress tensor as

$$\tilde{\tau}_{xx} = \frac{\tau_{xx}}{\varepsilon \rho_p g h_0}, \quad \tilde{\tau}_{zz} = \frac{\tau_{zz}}{\varepsilon \rho_p g h_0}, \quad \tilde{\tau}_{xz} = \frac{\tau_{xz}}{\rho_p g h_0} \quad (5.28)$$

and the traceless strain-rate tensor and energy as

$$|\tilde{\mathbf{S}}| = |\mathbf{S}| \frac{h_0}{u_0}, \quad \tilde{E} = \frac{E}{\rho_p u_0^2}. \quad (5.29)$$

We obtain for the dimensionless deviatoric stresses

$$\tilde{\tau}_{xx} = \mu(I) \frac{\tilde{p}}{|\tilde{\mathbf{S}}|} \left(\frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{w}}{\partial \tilde{z}} \right) = -\tilde{\tau}_{zz}, \quad \tilde{\tau}_{xz} = \mu(I) \frac{\tilde{p}}{|\tilde{\mathbf{S}}|} \left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \varepsilon^2 \frac{\partial \tilde{w}}{\partial \tilde{x}} \right). \quad (5.30)$$

with

$$I = \frac{|\tilde{\mathbf{S}}| \tilde{d}}{\sqrt{\tilde{p}}} F, \quad (5.31)$$

and

$$|\tilde{\mathbf{S}}| = \left[\left(\frac{\partial \tilde{u}}{\partial \tilde{z}} + \varepsilon^2 \frac{\partial \tilde{w}}{\partial \tilde{x}} \right)^2 + \varepsilon^2 \left(\frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{w}}{\partial \tilde{z}} \right)^2 \right]^{1/2}. \quad (5.32)$$

The dimensionless energy reads

$$\tilde{E} = \frac{1}{2} \phi (\tilde{u}^2 + \varepsilon^2 \tilde{w}^2) - \frac{\phi \sin \theta}{\varepsilon F^2} \tilde{x} + \frac{\phi \cos \theta}{F^2} \tilde{z}. \quad (5.33)$$

The equation of mass conservation stays true for the new variables:

$$\frac{\partial \phi}{\partial \tilde{t}} + \frac{\partial \phi \tilde{u}}{\partial \tilde{x}} + \frac{\partial \phi \tilde{w}}{\partial \tilde{z}} = 0. \quad (5.34)$$

The momentum equation in the x -direction becomes

$$\frac{\partial \phi \tilde{u}}{\partial \tilde{t}} + \frac{\partial \phi \tilde{u}^2}{\partial \tilde{x}} + \frac{\partial \phi \tilde{u} \tilde{w}}{\partial \tilde{z}} = \frac{\phi \sin \theta}{\varepsilon F^2} - \frac{1}{F^2} \frac{\partial \tilde{p}}{\partial \tilde{x}} + \frac{\varepsilon}{F^2} \frac{\partial \tilde{\tau}_{xx}}{\partial \tilde{x}} + \frac{1}{\varepsilon F^2} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}}. \quad (5.35)$$

In the z -direction, we obtain

$$\varepsilon^2 \left(\frac{\partial \phi \tilde{w}}{\partial \tilde{t}} + \frac{\partial \phi \tilde{u} \tilde{w}}{\partial \tilde{x}} + \frac{\partial \phi \tilde{w}^2}{\partial \tilde{z}} \right) = -\frac{\phi \cos \theta}{F^2} - \frac{1}{F^2} \frac{\partial \tilde{p}}{\partial \tilde{z}} + \frac{\varepsilon}{F^2} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{x}} + \frac{\varepsilon}{F^2} \frac{\partial \tilde{\tau}_{zz}}{\partial \tilde{z}}. \quad (5.36)$$

The energy equation (5.17) gives

$$\frac{\partial \tilde{E}}{\partial \tilde{t}} + \text{div}(\tilde{E} \tilde{\mathbf{v}}) = \frac{1}{F^2} \left[\tilde{u} \left(\varepsilon \frac{\partial \tilde{\tau}_{xx}}{\partial \tilde{x}} + \frac{1}{\varepsilon} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} - \frac{\partial \tilde{p}}{\partial \tilde{x}} \right) + \tilde{w} \left(\varepsilon \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{x}} + \varepsilon \frac{\partial \tilde{\tau}_{zz}}{\partial \tilde{z}} - \frac{\partial \tilde{p}}{\partial \tilde{z}} \right) \right], \quad (5.37)$$

where we denoted by $\tilde{\mathbf{v}}$ the dimensionless velocity field.

Finally, we can rewrite the boundary conditions. For the velocity, we obtain

$$\tilde{u}(0) = 0, \quad \tilde{w}(0) = 0, \quad \frac{\partial h}{\partial t} + \tilde{u}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}} = \tilde{w}(\tilde{h}). \quad (5.38)$$

Equations (5.23) become in the new variables

$$\left\{ \begin{array}{l} \tilde{\tau}_{xz}(\tilde{h}) + \varepsilon \tilde{p}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}} - \varepsilon^2 \tilde{\tau}_{xx}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}} = 0, \end{array} \right. \quad (5.39a)$$

$$\left\{ \begin{array}{l} \varepsilon \tilde{\tau}_{zz}(\tilde{h}) - \tilde{p}(\tilde{h}) - \varepsilon \tilde{\tau}_{xz}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}} = 0. \end{array} \right. \quad (5.39b)$$

3. Asymptotic expansion up to the first order

We now make the hypothesis that every variable f of the problem admits an asymptotic expansion as

$$f = f^{(0)} + \varepsilon f^{(1)} + O(\varepsilon^2). \quad (5.40)$$

We thus write such an expansion for every dimensionless variable of the problem. Injecting these expansions into the mass conservation law (5.34), the momentum balance equations (5.35)–(5.36) and the constitutive law (5.30), taking into account the boundary conditions, we can compute explicitly all terms in the expansions up to order 1.

3.1. Order 0

At leading order, we obtain from Equations (5.35), (5.36), (5.39) the following relations:

$$0 = \frac{\phi^{(0)} \sin \theta}{F^2} + \frac{1}{F^2} \frac{\partial \tilde{\tau}_{xz}^{(0)}}{\partial \tilde{z}}, \quad (5.41)$$

$$0 = -\frac{\phi^{(0)} \cos \theta}{F^2} - \frac{1}{F^2} \frac{\partial \tilde{p}^{(0)}}{\partial \tilde{z}} \quad (5.42)$$

$$0 = \tilde{\tau}_{xz}^{(0)}(\tilde{h}), \quad (5.43)$$

$$0 = \tilde{p}^{(0)}(\tilde{h}). \quad (5.44)$$

We integrate the first two equations between $\tilde{z}$ and $\tilde{h}$ using the conditions at $\tilde{z} = \tilde{h}$ to obtain

$$\tilde{\tau}_{xz}^{(0)} = \sin \theta \int_{\tilde{z}}^{\tilde{h}} \phi^{(0)} d\tilde{z} \quad (5.45)$$

and

$$\tilde{p}^{(0)} = \cos \theta \int_{\tilde{z}}^{\tilde{h}} \phi^{(0)} d\tilde{z}. \quad (5.46)$$

5. *Modèle consistant d'écoulement granulaire compressible – 3. Asymptotic expansion up to the first order*

Equation (5.32) gives, after using equation (5.30) implying that $\partial \tilde{u}^{(0)} / \partial \tilde{z} > 0$,

$$|\tilde{\mathbf{S}}|^{(0)} = \frac{\partial \tilde{u}^{(0)}}{\partial \tilde{z}}. \quad (5.47)$$

Equation (5.30) then gives

$$\tilde{\tau}_{xz}^{(0)} = \mu^{(0)} \tilde{p}^{(0)} \quad (5.48)$$

with $\mu^{(0)} = \mu(I^{(0)})$. We have $\mu^{(0)} = \tau_{xz}^{(0)} / p^{(0)}$. This yields

$$\mu^{(0)} = \tan \theta. \quad (5.49)$$

At order 0, the friction coefficient has the same expression as in the case of a steady uniform flow. The expression of the inertial number at order 0 is

$$I^{(0)} = \frac{\tan \theta - \mu_1}{\mu_2 - \tan \theta} I_m \quad (5.50)$$

by (5.10). Hence the inertial number at order 0, $I^{(0)}$, is a constant. It follows that it is also the case for the volume fraction:

$$\phi^{(0)} = \phi(I^{(0)}) \quad (5.51)$$

is constant. We deduce that

$$\tilde{\tau}_{xz}^{(0)} = \phi^{(0)} \sin \theta (\tilde{h} - \tilde{z}) \quad (5.52)$$

and

$$\tilde{p}^{(0)} = \phi^{(0)} \cos \theta (\tilde{h} - \tilde{z}). \quad (5.53)$$

The pressure is thus hydrostatic at this order. We also obtain from Equation (5.34) that, at order 0, the flow is incompressible:

$$\frac{\partial \tilde{u}^{(0)}}{\partial \tilde{x}} + \frac{\partial \tilde{w}^{(0)}}{\partial \tilde{z}} = 0. \quad (5.54)$$

As a consequence, the formulas obtained in the context of the incompressible rheology (see for instance the work of Deleage and Richard 2025) are still valid at order 0 for this version of the compressible $\mu(I)$ -rheology. We deduce that

$$\tilde{u}^{(0)} = \frac{2\tilde{A}}{3} \left[\tilde{h}^{3/2} - (\tilde{h} - \tilde{z})^{3/2} \right], \quad (5.55)$$

where we defined

$$\tilde{A} = \frac{I^{(0)}}{\tilde{d}F} \sqrt{\phi^{(0)} \cos \theta}, \quad \text{and} \quad A = \frac{I^{(0)} \sqrt{\phi^{(0)} g \cos \theta}}{d}. \quad (5.56)$$

At order 0, the velocity profile in the z -direction is a Bagnold profile. At this order, the depth-averaged velocity is $\tilde{U}^{(0)} = 2\tilde{A}\tilde{h}^{3/2}/5$. If the characteristic velocity u_0 and the vertical scale h_0 are taken equal respectively to the depth-averaged velocity and depth of the steady uniform flow, then $\tilde{A} = 5/2$. We also have

$$\tilde{w}^{(0)} = \tilde{A} \left[\frac{2}{3} \tilde{h}^{3/2} - \frac{2}{3} (\tilde{h} - \tilde{z})^{3/2} - \tilde{z} \tilde{h}^{1/2} \right] \frac{\partial \tilde{h}}{\partial \tilde{x}}. \quad (5.57)$$

and

$$\tilde{\tau}_{xx}^{(0)} = -\tilde{\tau}_{zz}^{(0)} = 2\phi^{(0)} \sin \theta \left[\tilde{h}^{1/2} (\tilde{h} - \tilde{z})^{1/2} - \tilde{h} + \tilde{z} \right] \frac{\partial \tilde{h}}{\partial \tilde{x}}. \quad (5.58)$$

We refer to the work of Deleage and Richard 2025 for detailed computations.

3.2. Order 1

We now investigate Equations (5.35), (5.36) and (5.39) at the first order. At this order of accuracy, these equations become

$$\frac{\partial \tilde{u}^{(0)}}{\partial \tilde{t}} + \tilde{u}^{(0)} \frac{\partial \tilde{u}^{(0)}}{\partial \tilde{x}} + \tilde{w}^{(0)} \frac{\partial \tilde{u}^{(0)}}{\partial \tilde{z}} = \frac{\phi^{(1)} \sin \theta}{\phi^{(0)} F^2} - \frac{1}{\phi^{(0)} F^2} \frac{\partial \tilde{p}^{(0)}}{\partial \tilde{x}} + \frac{1}{\phi^{(0)} F^2} \frac{\partial \tilde{\tau}_{xz}^{(1)}}{\partial \tilde{z}}, \quad (5.59)$$

$$0 = -\phi^{(1)} \cos \theta - \frac{\partial \tilde{p}^{(1)}}{\partial \tilde{z}} + \frac{\partial \tilde{\tau}_{xz}^{(0)}}{\partial \tilde{x}} + \frac{\partial \tilde{\tau}_{zz}^{(0)}}{\partial \tilde{z}} \quad (5.60)$$

$$0 = \tilde{\tau}_{xz}^{(1)}(\tilde{h}) + \tilde{p}^{(0)}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}}, \quad (5.61)$$

$$0 = \tilde{\tau}_{zz}^{(0)}(\tilde{h}) - \tilde{p}^{(1)}(\tilde{h}) - \tilde{\tau}_{xz}^{(0)}(\tilde{h}) \frac{\partial \tilde{h}}{\partial \tilde{x}}, \quad (5.62)$$

where we used the fact that the flow is incompressible at order 0 to rewrite the conservative term of (5.59) as a transport term. In view of the values of $\tilde{\tau}_{xz}^{(0)}(\tilde{h})$, $\tilde{\tau}_{zz}^{(0)}(\tilde{h})$ and $\tilde{p}^{(0)}(\tilde{h})$ (see (5.52), (5.58) and (5.53)), we obtain from the two last equations

$$\tilde{\tau}_{xz}^{(1)}(\tilde{h}) = \tilde{p}^{(1)}(\tilde{h}) = 0. \quad (5.63)$$

We inject the values of $\tilde{u}^{(0)}$, $\tilde{w}^{(0)}$ and $\tilde{p}^{(0)}$, using Equation (5.38) to express $\partial \tilde{h} / \partial \tilde{t}$ at the leading order as $\partial \tilde{h} / \partial \tilde{t} = -\tilde{A} \tilde{h}^{3/2} \partial \tilde{h} / \partial \tilde{x}$, to obtain

$$\frac{\partial \tilde{\tau}_{xz}^{(1)}}{\partial \tilde{z}} = \phi^{(0)} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\cos \theta + \frac{F^2 \tilde{A}^2}{3} (\tilde{h}^{1/2} (\tilde{h} - \tilde{z})^{3/2} - \tilde{h}^2) \right] - \phi^{(1)} \sin \theta. \quad (5.64)$$

We now inject the values of $\tilde{\tau}_{xz}^{(0)}$ and $\tilde{\tau}_{zz}^{(0)}$ into the second equation. We find

$$\frac{\partial \tilde{p}^{(1)}}{\partial \tilde{z}} = \phi^{(0)} \sin \theta \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\frac{\tilde{h}^{1/2}}{(\tilde{h} - \tilde{z})^{1/2}} - 1 \right] - \phi^{(1)} \cos \theta. \quad (5.65)$$

We thus need to compute $\phi^{(1)}$. The relations $\phi = \phi(I)$ and $I = I(\mu)$ (see (5.10)) imply

that ϕ can be seen as a function of μ only. See for instance Fall et al. 2015 for an experimental study of the relation $\phi = \phi(\mu)$. It follows that

$$\phi^{(1)} = -D\phi^{(0)}\mu^{(1)}, \quad (5.66)$$

where $D = -(\partial \ln \phi / \partial \mu)(\mu = \tan \theta)$ or

$$D = -\frac{1}{\phi^{(0)}} \frac{\partial \phi}{\partial I} (I = I^{(0)}) I_m \frac{\mu_2 - \mu_1}{(\mu_2 - \tan \theta)^2}. \quad (5.67)$$

For instance, if the affine law (5.12) is used, one obtains for D the following value:

$$D = \frac{\phi_c - \phi_m}{\phi^{(0)}} I_m \frac{\mu_2 - \mu_1}{(\mu_2 - \tan \theta)^2}. \quad (5.68)$$

Similarly, one can deduce the expression of the first-order correction to the inertial number, writing

$$I^{(1)} = \left. \frac{\partial I}{\partial \mu} \right|_{\mu^{(0)}} \mu^{(1)}, \quad (5.69)$$

to find

$$I^{(1)} = I_m \frac{\mu_2 - \mu_1}{(\mu_2 - \tan \theta)^2} \mu^{(1)}. \quad (5.70)$$

We can write

$$I^{(1)} = B I^{(0)} \mu^{(1)} \quad (5.71)$$

where $B = (\partial \ln I / \partial \mu)(\mu = \tan \theta)$ or

$$B = \frac{\mu_2 - \mu_1}{(\mu_2 - \tan \theta)(\tan \theta - \mu_1)}. \quad (5.72)$$

We also obtain from (5.32)

$$|\tilde{\mathbf{S}}|^{(1)} = \frac{\partial \tilde{u}^{(1)}}{\partial \tilde{z}}. \quad (5.73)$$

By Equations (5.30) and (5.47), (5.73), we also have that

$$\tilde{\tau}_{xz}^{(1)} = \mu^{(1)} \tilde{p}^{(0)} + \mu^{(0)} \tilde{p}^{(1)}. \quad (5.74)$$

We differentiate this last relation with respect to $\tilde{z}$ (recall that $\mu^{(0)}$ is constant):

$$\frac{\partial \tilde{\tau}_{xz}^{(1)}}{\partial \tilde{z}} = \frac{\partial}{\partial \tilde{z}} (\tilde{p}^{(0)} \mu^{(1)}) + \mu^{(0)} \frac{\partial \tilde{p}^{(1)}}{\partial \tilde{z}}.$$

By injecting the values of $\tilde{p}^{(0)}$ and the $\tilde{z}$ -derivatives of $\tilde{p}^{(1)}$ and $\tilde{\tau}_{xz}^{(1)}$ that we computed before, we obtain that $\mu^{(1)}$ solves

$$\frac{\partial}{\partial \tilde{z}} [(\tilde{h} - \tilde{z}) \mu^{(1)}] = \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[1 + \tan^2 \theta + \frac{F^2 A^2}{3 \cos \theta} (\tilde{h}^{1/2} (\tilde{h} - \tilde{z})^{3/2} - \tilde{h}^2) - \frac{\tan^2 \theta \tilde{h}^{1/2}}{(\tilde{h} - \tilde{z})^{1/2}} \right].$$

5. Modèle consistant d'écoulement granulaire compressible – 3. Asymptotic expansion up to the first order

We integrate between $\tilde{z}$ and $\tilde{h}$ (note that (5.74) and (5.63) implies that $(\tilde{p}^{(0)}\mu^{(1)})(\tilde{h}) = 0$) to get that the first-order correction to the friction coefficient can be written as

$$\mu^{(1)} = \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\frac{F^2 A^2}{3 \cos \theta} \left(\tilde{h}^2 - \frac{2\tilde{h}^{1/2}}{5} (\tilde{h} - \tilde{z})^{3/2} \right) + \frac{2 \tan^2 \theta \tilde{h}^{1/2}}{(\tilde{h} - \tilde{z})^{1/2}} - \frac{1}{\cos^2 \theta} \right], \quad (5.75)$$

and, with (5.66),

$$\frac{\phi^{(1)}}{\phi^{(0)}} = -D\mu^{(1)} = -D \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\frac{F^2 A^2}{3 \cos \theta} \left(\tilde{h}^2 - \frac{2\tilde{h}^{1/2}}{5} (\tilde{h} - \tilde{z})^{3/2} \right) + \frac{2 \tan^2 \theta \tilde{h}^{1/2}}{(\tilde{h} - \tilde{z})^{1/2}} - \frac{1}{\cos^2 \theta} \right]. \quad (5.76)$$

We deduce that

$$\begin{aligned} \int_{\tilde{z}}^{\tilde{h}} \phi^{(1)} = & -\phi^{(0)} D \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\frac{F^2 A^2}{3 \cos \theta} \left(\tilde{h}^2 (\tilde{h} - \tilde{z}) - \frac{4\tilde{h}^{1/2}}{25} (\tilde{h} - \tilde{z})^{5/2} \right) \right. \\ & \left. + 4 \tan^2 \theta \tilde{h}^{1/2} (\tilde{h} - \tilde{z})^{1/2} - \frac{\tilde{h} - \tilde{z}}{\cos^2 \theta} \right]. \end{aligned} \quad (5.77)$$

We can now integrate (5.64) between $\tilde{z}$ and $\tilde{h}$ to obtain

$$\begin{aligned} \tilde{\tau}_{xz}^{(1)} = & \phi^{(0)} \frac{\partial \tilde{h}}{\partial x} \left[-4D \sin \theta \tan^2 \theta \tilde{h}^{1/2} (\tilde{h} - \tilde{z})^{1/2} - (\tilde{h} - \tilde{z}) \left(\cos \theta - \frac{D \sin \theta}{\cos^2 \theta} \right) \right. \\ & \left. + \frac{F^2 A^2}{3} \left(\tilde{h}^2 (\tilde{h} - \tilde{z}) (1 - D \tan \theta) - \frac{2}{5} \tilde{h}^{1/2} (\tilde{h} - \tilde{z})^{5/2} (1 - \frac{2}{5} D \tan \theta) \right) \right] \end{aligned} \quad (5.78)$$

We now use (5.74) to calculate the expression of the first-order correction to the pressure as $\tilde{p}^{(1)} = (\tilde{\tau}_{xz}^{(1)} - \mu^{(1)} \tilde{p}^{(0)}) / \mu^{(0)}$, i.e.

$$\begin{aligned} \tilde{p}^{(1)} = & \phi^{(0)} \frac{\partial \tilde{h}}{\partial x} \left[\left(-4D - \frac{2}{\tan \theta} \right) \frac{\sin^2 \theta}{\cos \theta} \tilde{h}^{1/2} (\tilde{h} - \tilde{z})^{1/2} - (\tilde{h} - \tilde{z}) \left(\frac{-D}{\cos \theta} - \sin \theta \right) \right. \\ & \left. - \frac{DF^2 A^2}{3} \left(\tilde{h}^2 (\tilde{h} - \tilde{z}) - \frac{4}{25} \tilde{h}^{1/2} (\tilde{h} - \tilde{z})^{5/2} \right) \right]. \end{aligned} \quad (5.79)$$

Equations (5.79), (5.71), (5.73) and (5.53) can be used to obtain

$$\begin{aligned} \frac{\partial \tilde{u}^{(1)}}{\partial \tilde{z}} = & A \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\left(-2D + 2B - \frac{1}{\tan \theta} \right) \tan^2 \theta \tilde{h}^{1/2} - \frac{(\tilde{h} - \tilde{z})^{1/2}}{2} \left(\frac{-D}{\cos^2 \theta} + \frac{2B}{\cos^2 \theta} - \tan \theta \right) \right. \\ & \left. - \frac{DF^2 A^2}{6 \cos \theta} \left(\tilde{h}^2 (\tilde{h} - \tilde{z})^{1/2} - \frac{4}{25} \tilde{h}^{1/2} (\tilde{h} - \tilde{z})^2 \right) + B \frac{F^2 A^2}{3 \cos \theta} \left(\tilde{h}^2 (\tilde{h} - \tilde{z})^{1/2} - \frac{2\tilde{h}^{1/2}}{5} (\tilde{h} - \tilde{z})^2 \right) \right]. \end{aligned}$$

We integrate this expression between 0 and $\tilde{z}$ and obtain that

$$\begin{aligned} \tilde{u}^{(1)} = & A \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\left(-2D + 2B - \frac{1}{\tan \theta} \right) \tan^2 \theta \tilde{h}^{1/2} \tilde{z} - \frac{\tilde{h}^{3/2} - (\tilde{h} - \tilde{z})^{3/2}}{3} \left(\frac{-D}{\cos^2 \theta} + \frac{2B}{\cos^2 \theta} - \tan \theta \right) \right. \\ & - \frac{DF^2 A^2}{9 \cos \theta} \left(\tilde{h}^2 (\tilde{h}^{3/2} - (\tilde{h} - \tilde{z})^{3/2}) - \frac{2\tilde{h}^3 - 2(\tilde{h} - \tilde{z})^3}{25} \tilde{h}^{1/2} \right) \\ & \left. + B \frac{2F^2 A^2}{9 \cos \theta} \left(\tilde{h}^2 (\tilde{h}^{3/2} - (\tilde{h} - \tilde{z})^{3/2}) - \frac{\tilde{h}^3 - (\tilde{h} - \tilde{z})^3}{5} \tilde{h}^{1/2} \right) \right]. \end{aligned} \quad (5.80)$$

which is the consistent first-order correction to the Bagnold velocity profile.

4. Derivation of the depth-averaged equations

4.1. Averaged variables

We now define depth-averaged variables, and give an explicit asymptotic expansion of these variables up to the first order. The total mass over the depth m is defined by

$$m = \int_0^h \rho dz. \quad (5.81)$$

This quantity is sometimes also called the "mass hold-up" in the literature (see for instance P. C. Johnson et al. 1990; Louge et al. 2001; Holyoake et al. 2012). It enables to define the averaged volume fraction by

$$\bar{\phi} = \frac{1}{h} \int_0^h \phi dz = \frac{m}{\rho_p h}. \quad (5.82)$$

We see that the variable m has an important physical meaning in the compressible case. Indeed, to the same value of h can correspond various values of $\bar{\phi}$. The mass hold-up is scaled by $m = \tilde{m} \rho_p h_0$. It follows that

$$\tilde{m}^{(0)} = \phi^{(0)} \tilde{h} \quad (5.83)$$

and

$$\tilde{m}^{(1)} = -\phi^{(0)} D \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{7F^2 A^2}{25 \cos \theta} \tilde{h}^3 + (3 \tan^2 \theta - 1) \tilde{h} \right). \quad (5.84)$$

In order to simplify the equations, we also introduced the normalised hold-up as

$$r = \int_0^h \frac{\rho}{\rho^{(0)}} dz, \quad \text{with} \quad \rho^{(0)} = \rho_p \phi^{(0)}. \quad (5.85)$$

5. *Modèle consistant d'écoulement granulaire compressible – 4. Derivation of the depth-averaged equations*

It follows that $m/\rho_p = \phi^{(0)} r = \bar{\phi} h$. r has the dimension of a length and is scaled as $r = h_0 \tilde{r}$. We have that

$$\tilde{r}^{(0)} = \tilde{h} \quad (5.86)$$

and

$$\tilde{r}^{(1)} = -D \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{7F^2 A^2}{25 \cos \theta} \tilde{h}^3 + (3 \tan^2 \theta - 1) \tilde{h} \right). \quad (5.87)$$

The Favre averaging over the depth of any quantity X is then defined as

$$\langle X \rangle = \frac{1}{m} \int_0^h \rho X \, dz = \frac{1}{r} \int_0^h \frac{\rho}{\rho^{(0)}} X \, dz = \frac{1}{r} \int_0^h \frac{\phi}{\phi^{(0)}} X \, dz. \quad (5.88)$$

To lighten the notation, the depth-averaged velocity is denoted by $U = \langle u \rangle$. The average velocity $\tilde{U}$ admits an asymptotic expansion given by

$$\tilde{U} = \tilde{U}^{(0)} + \varepsilon \tilde{U}^{(1)} + O(\varepsilon^2), \quad (5.89)$$

where, by (5.55),

$$\tilde{U}^{(0)} = \frac{2}{5} \tilde{A} \tilde{h}^{3/2}, \quad (5.90)$$

and, from (5.80), after some computations,

$$\tilde{U}^{(1)} = \frac{\partial \tilde{h}}{\partial \tilde{x}} \frac{F^2 A^3}{25 \cos \theta} A_1 \tilde{h}^{7/2} + \frac{A}{5} \frac{\partial \tilde{h}}{\partial \tilde{x}} A_2 \tilde{h}^{3/2}, \quad (5.91)$$

with

$$A_1 = -\frac{17}{10} D + \frac{5}{2} B \quad (5.92)$$

and

$$A_2 = -(6 \tan^2 \theta - 1) D + B(3 \tan^2 \theta - 2) - \frac{3}{2} \tan \theta, \quad (5.93)$$

Note that from the expressions of $\tilde{U}^{(0)}$ and $\tilde{r}^{(0)}$, one can deduce the relation

$$\tilde{U}^{(0)} = \frac{2}{5} A (\tilde{r}^{(0)})^{3/2}. \quad (5.94)$$

Hence we obtain the following expansion:

$$\tilde{U} - \frac{2}{5} A \tilde{r}^{3/2} = \varepsilon \left(\tilde{U}^{(1)} - \frac{3}{5} A \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) + O(\varepsilon^2). \quad (5.95)$$

We compute that the first-order term in (5.95) is given by

$$\tilde{U}^{(1)} - \frac{3}{5} A \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} = \frac{\tilde{A}(B+D)}{10} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\frac{F^2 \tilde{A}^2}{\cos \theta} \tilde{h}^{7/2} + (3C-4) \tilde{h}^{3/2} \right], \quad (5.96)$$

5. *Modèle consistant d'écoulement granulaire compressible – 4. Derivation of the depth-averaged equations*

where, to simplify notations, we defined

$$C = \tan \theta \left(2 \tan \theta - \frac{1}{B+D} \right). \quad (5.97)$$

Note that $C = C(\theta)$ is a positive function of θ , which takes values in the interval $[2\mu_1^2, 2\mu_2^2]$ if $\theta \in [\arctan \mu_1, \arctan \mu_2]$. Indeed, since by assumption $D > 0$, we have the following bounds for C :

$$\tan^2 \theta + \tan \theta \left[\frac{(\tan \theta - \mu_1)^2}{\mu_2 - \mu_1} + \mu_1 \right] = \tan \theta \left(2 \tan \theta - \frac{1}{B} \right) < C < 2 \tan^2 \theta. \quad (5.98)$$

The definition of the constant C can be seen as a generalisation of the one given by Deleage and Richard 2025 in the incompressible case. Indeed, it reduces to that definition (see equation (4.4) of chapter 3) when ϕ is constant, i.e. when $D = 0$ (see (5.67)). When $D = 0$, the expression of C corresponds to the left-hand side of (5.98). It is also interesting to note that the expression (5.96) was already obtained in the incompressible case. Surprisingly, the first order correction induced by the introduction of dilatancy in the constitutive equations can be taken into account in a very simple way. Indeed, it is enough to use the expression obtained in the incompressible case (see equation (3.78) of chapter 3) and replace the constant B by the modified version $B + D$.

Note that the first-order correction (5.95) can be written

$$\tilde{U}^{(1)} - \frac{3}{5} A \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} = \frac{1}{\varepsilon} \left(\tilde{U} - \frac{2}{5} \tilde{A} \tilde{r}^{3/2} \right) + O(\varepsilon), \quad (5.99)$$

We also define a quantity, called enstrophy, as $\psi = \langle (u - U)^2 \rangle / r^2$. The enstrophy takes shear effects into account. It is zero for a velocity constant in the depth and positive otherwise. The enstrophy is scaled as $\psi = \tilde{\psi} u_0^2 / h_0^2$. With this definition, $\tilde{r} \langle \tilde{u}^2 \rangle = \tilde{r} \tilde{U}^2 + \tilde{r}^3 \tilde{\psi}$. Similarly, $\tilde{\psi}$ expands as $\tilde{\psi} = \tilde{\psi}^{(0)} + \varepsilon \tilde{\psi}^{(1)} + O(\varepsilon^2)$, where

$$\tilde{\psi}^{(0)} = \frac{\tilde{A}^2 \tilde{h}}{25} \quad (5.100)$$

is the consistent expression of the enstrophy for a Bagnold profile, and

$$\tilde{\psi}^{(1)} = \frac{\partial \tilde{h}}{\partial \tilde{x}} \frac{F^2 A^3}{25 \cos \theta} A_3 \tilde{h}^3 + \frac{\partial \tilde{h}}{\partial \tilde{x}} \frac{A}{5} A_4 \tilde{h}, \quad (5.101)$$

with

$$A_3 := \tilde{A} \left[\frac{73}{275} D + \frac{6}{11} B \right] \quad (5.102)$$

and

$$A_4 := \tilde{A} \left[\left(\frac{1}{7} \tan^2 \theta - \frac{1}{5} \right) D + B \left(\frac{13}{35} \tan \theta \left(2 \tan \theta - \frac{1}{B} \right) - \frac{2}{5} \right) \right]. \quad (5.103)$$

At leading order, the enstrophy and the depth-averaged velocity satisfy the relation

$$\tilde{\psi}^{(0)} = \frac{(\tilde{U}^{(0)})^2}{4(\tilde{r}^{(0)})^2}. \quad (5.104)$$

Note that this expression gives at leading order $r\langle(u^{(0)})^2\rangle = (5/4)r(U^{(0)})^2$. This is consistent with the shape factor 5/4 which is obtained for a Bagnold velocity profile.

The relation (5.104) leads to the expansion

$$\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{r}^2} = \epsilon \left(\psi^{(1)} - \frac{\tilde{U}^{(0)}\tilde{U}^{(1)}}{2(\tilde{r}^{(0)})^2} + \frac{\tilde{r}^{(1)}(\tilde{U}^{(0)})^2}{2(\tilde{r}^{(0)})^3} \right) + O(\epsilon^2). \quad (5.105)$$

The leading term of (5.105) can be computed with Equations (5.101) and (5.96). We obtain

$$\tilde{\psi}^{(1)} - \frac{\tilde{U}^{(0)}\tilde{U}^{(1)}}{2(\tilde{r}^{(0)})^2} + \frac{\tilde{r}^{(1)}(\tilde{U}^{(0)})^2}{2(\tilde{r}^{(0)})^3} = \frac{\tilde{A}^2(B+D)}{10} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{F^2 \tilde{A}^2}{55 \cos \theta} \tilde{h}^3 + \frac{C}{7} \tilde{h} \right). \quad (5.106)$$

This expression is useful to write consistently a relaxation term for the enstrophy. Again, the only difference with the incompressible case is that the constant B of equation (3.83) of chapter 3 has been replaced by $B+D$ in the right-hand side of (5.106).

4.2. Depth-averaged mass and momentum equations

We now want to derive evolution equations for the depth-averaged variables. We first integrate the mass conservation (5.34) between 0 and $\tilde{h}$. Using the kinematic boundary condition (5.38), we obtain the depth-averaged mass conservation equation

$$\frac{\partial \tilde{m}}{\partial \tilde{t}} + \frac{\partial \tilde{m} \tilde{U}}{\partial \tilde{x}} = 0, \quad (5.107)$$

which can also be written with the variable r as

$$\frac{\partial \tilde{r}}{\partial \tilde{t}} + \frac{\partial \tilde{r} \tilde{U}}{\partial \tilde{x}} = 0. \quad (5.108)$$

In order to derive an equation for $\tilde{U}$, we integrate (5.35) between 0 and $\tilde{h}$ using the boundary condition (5.38). This leads to a preliminary expression of the depth-averaged momentum balance equation

$$\frac{\partial \tilde{r} \tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\tilde{r} \tilde{U}^2 + \tilde{r}^3 \tilde{\psi} + \frac{1}{\phi^{(0)} F^2} \int_0^{\tilde{h}} \tilde{p} d\tilde{z} \right) = \frac{1}{\epsilon \phi^{(0)} F^2} (\phi^{(0)} \tilde{r} \sin \theta \tilde{h} - \tilde{\tau}_{xz}(0)) + O(\epsilon). \quad (5.109)$$

5. *Modèle consistant d'écoulement granulaire compressible – 4. Derivation of the depth-averaged equations*

We first compute the integral of the pressure $\tilde{p}$. By using the expression (5.53) of $\tilde{p}^{(0)}$, we obtain

$$\frac{1}{\phi^{(0)}} \int_0^{\tilde{h}} \tilde{p} d\tilde{z} = \frac{1}{\phi^{(0)}} \int_0^{\tilde{h}} \tilde{p}^{(0)} d\tilde{z} + O(\varepsilon) = \frac{\tilde{r}^2 \cos \theta}{2} + O(\varepsilon). \quad (5.110)$$

Equation (5.109) thus yields

$$\frac{\partial \tilde{r} \tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\tilde{r} \tilde{U}^2 + \tilde{r}^3 \tilde{\psi} + \frac{\tilde{r}^2 \cos \theta}{2F^2} \right) = \frac{1}{\varepsilon \phi^{(0)} F^2} (\phi^{(0)} \tilde{r} \sin \theta - \tilde{\tau}_{xz}(0)) + O(\varepsilon). \quad (5.111)$$

We now need to find an expression for the basal friction $\tilde{\tau}_{xz}(0)$. For an incompressible flow with constant volume fraction, the basal friction can be written $\tilde{\tau}_{xz}(0) = \mu_{b,i} \phi \tilde{h} \cos \theta$ (Andreotti et al. 2013) where

$$\mu_{b,i} = \mu_1 + \frac{\mu_2 - \mu_1}{I_m \frac{2\tilde{h} \sqrt{\phi \tilde{h} \cos \theta}}{5\tilde{d}F\tilde{U}} + 1}. \quad (5.112)$$

This expression cannot be used directly in the compressible case. Indeed, whereas the volume fraction ϕ is constant for an incompressible flow, it is not true anymore in the compressible case (see (5.66) for instance). The letter ϕ is thus ambiguous here, as it could correspond to the volume fraction at equilibrium $\phi^{(0)}$, to the averaged volume fraction $\bar{\phi}$ or to the volume fraction at the bottom $\phi(0)$ for instance. Hence some choices have to be made here. For a compressible granular flow, we propose the following generalisation:

$$\mu_{b,i} \phi \tilde{h} \cos \theta \rightarrow \xi \mu_b \phi^{(0)} \tilde{r} \cos \theta, \quad (5.113)$$

where

$$\mu_b = \mu_1 + \frac{\mu_2 - \mu_1}{I_m \frac{2\tilde{r} \sqrt{\phi^{(0)} \tilde{r} \cos \theta}}{5\tilde{d}F\tilde{U}} + 1} \quad (5.114)$$

is the generalisation of the friction law $\mu_{b,i}$ and

$$\xi = \exp \left[-\frac{5D}{2\tilde{A}\tilde{r}^{3/2}B(B+D)\tan \theta} \left(U - \frac{2\tilde{A}}{5}\tilde{r}^{3/2} \right) \right] \quad (5.115)$$

is a factor modelling dilatancy. Let us make a few qualitative comments in order to motivate these choices:

- The product $\phi \tilde{h} \cos \theta$, corresponding to the dimensionless basal normal stress in the incompressible case, is replaced by $\phi^{(0)} \tilde{r} \cos \theta = \tilde{m} \cos \theta$ in the compressible case, both in factor of the friction law and inside the expression of μ_b .
- The remaining dependence on $\tilde{h}$ of the friction law $\mu_{b,i}$ is removed by replacing $\tilde{h}$ by $\tilde{r}$. This means that the basal friction μ_b only depends on the velocity and

5. *Modèle consistant d'écoulement granulaire compressible – 4. Derivation of the depth-averaged equations*

on the mass hold-up, corresponding to the (normalised) mass of the granular column. Recall that at equilibrium, it holds that $\bar{\phi} = \phi^{(0)}$, hence $\tilde{r} = \tilde{h}$ and we recover the friction law of Andreotti et al. 2013.

- The factor ξ is taking dilatancy into account. For an incompressible flow ($D = 0$) or at equilibrium ($\tilde{U} = \tilde{U}^{(0)}$ and $\tilde{r} = \tilde{r}^{(0)}$), it is equal to 1. When the mean velocity is greater than the velocity at equilibrium, $\xi < 1$. On the contrary, when U is smaller than its value at equilibrium, $\xi > 1$. It follows that the increasing of the basal friction with the mean velocity is still true in the compressible case, but is slightly attenuated compared to the incompressible case, because of this corrective factor. Note that ξ is only a first-order correction to the friction law, hence this attenuation effect stays small close to stationary and uniform flows. As it is shown hereafter, the exponential function appearing in (5.115) could be replaced by any function f such that $f(0) = 1$ and $f'(0) = 1$, without any loss of consistency. This means that close to a stationary uniform flow, any choice of f with $f(0) = f'(0) = 1$ would yield similar results. However, changing the function f could drastically affect the behaviour far from stationary uniform flows. We choose here $f = \exp$ to guarantee the positivity of the modified friction coefficient, but other choices are possible and a further study, including comparison with experimental data in non stationary and non uniform regimes, would be needed to obtain an expression of ξ which stays valid far from steady uniform flows.

Let us now justify mathematically the formula (5.113). We will show by an exact expansion up to order 1 that it enables to recover the same structure that was derived in the incompressible case by Deleage and Richard 2025. The expression (5.114) can be written

$$\mu_b = \mu_1 + \frac{\mu_2 - \mu_1}{\frac{I_m}{I^{(0)}} \left(\frac{\tilde{r}}{\tilde{r}^{(0)}} \right)^{3/2} \frac{\tilde{U}^{(0)}}{\tilde{U}} + 1} \quad (5.116)$$

which means that $\mu_b = \mu[I = I^{(0)}(\tilde{r}^{(0)}/\tilde{r})^{3/2}\tilde{U}/\tilde{U}^{(0)}]$ such that $\mu_b^{(0)} = \mu^{(0)}$. The expansion of μ_b gives

$$\mu_b = \mu^{(0)} + \varepsilon \left. \frac{\partial \mu}{\partial I} \right|_{I^{(0)}} \frac{I^{(0)}}{\tilde{U}^{(0)}} \left(\tilde{U}^{(1)} - \frac{3}{5} \tilde{A} \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) + O(\varepsilon^2) \quad (5.117)$$

which can be written as

$$\mu_b = \mu^{(0)} + \varepsilon \frac{1}{B \tilde{U}^{(0)}} \left(\tilde{U}^{(1)} - \frac{3}{5} \tilde{A} \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) + O(\varepsilon^2). \quad (5.118)$$

Similarly, the factor ξ can be written as

$$\xi = \exp \left[- \frac{D}{\tilde{U}^{(0)} B (B + D) \mu^{(0)}} \left(\tilde{U} \left(\frac{\tilde{r}^{(0)}}{\tilde{r}} \right)^{3/2} - \tilde{U}^{(0)} \right) \right], \quad (5.119)$$

5. *Modèle consistant d'écoulement granulaire compressible – 4. Derivation of the depth-averaged equations*

which yields

$$\xi = 1 - \varepsilon \frac{D}{\tilde{U}^{(0)} B(B+D) \mu^{(0)}} \left(\tilde{U}^{(1)} - \frac{3}{5} \tilde{A} \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) + O(\varepsilon^2). \quad (5.120)$$

Note that the expansion (5.120) still holds if the exponential function appearing in the definition of ξ is replaced by any function f with $f(0) = f'(0) = 1$. The product $\xi \mu_b$ can be expanded as

$$\xi \mu_b = \xi^{(0)} \mu_b^{(0)} + \varepsilon \left(\xi^{(1)} \mu_b^{(0)} + \xi^{(0)} \mu_b^{(1)} \right) + O(\varepsilon^2), \quad (5.121)$$

which yields

$$\xi \mu_b = \mu^{(0)} + \varepsilon \frac{1}{(B+D) \tilde{U}^{(0)}} \left(\tilde{U}^{(1)} - \frac{3}{5} \tilde{A} \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) + O(\varepsilon^2). \quad (5.122)$$

In particular, we get that $(\xi \mu_b)^{(0)} = \mu^{(0)}$, which means that the generalized friction law is consistent at order 0 with the $\mu(I)$ -rheology (i.e. $\tilde{\tau}_{xz}^{(0)}(0) = \phi^{(0)}(\xi \mu_b \tilde{r})^{(0)} \cos \theta$). However, it is not the case at order one: $\tilde{\tau}_{xz}^{(1)}(0) \neq \phi^{(0)}(\xi \mu_b \tilde{r})^{(1)} \cos \theta$. Indeed, using Equations (5.52), (5.53) and (5.78), the shear stress at the bottom can be written

$$\tilde{\tau}_{xz}(0) = \mu^{(0)} \phi^{(0)} \tilde{r}^{(0)} \cos \theta + \varepsilon \phi^{(0)} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{F^2 \tilde{A}^2}{5} \tilde{h}^3 - \tilde{h} \cos \theta \right) + \varepsilon \phi^{(0)} \tilde{r}^{(1)} \sin \theta + O(\varepsilon^2). \quad (5.123)$$

Subtracting $\xi \mu_b \phi^{(0)} \tilde{r} \cos \theta$ from $\tilde{\tau}_{xz}(0)$ leads to the expression of the first-order correction to the friction law

$$\tilde{\tau}_{xz}(0) - \xi \mu_b \phi^{(0)} \tilde{r} \cos \theta = \varepsilon \phi^{(0)} R \cos \theta + O(\varepsilon^2), \quad (5.124)$$

where

$$R = -\frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{F^2 \tilde{A}^2}{20 \cos \theta} \tilde{h}^3 + \frac{3C}{4} \tilde{h} \right). \quad (5.125)$$

We see that the term R obtained is exactly the same as the one obtained in the incompressible case (Deleage and Richard 2025). This means that all the dilatancy effects coming from the rheology are encoded by the replacement of the variable h by r , the replacement of the constant B by $B+D$ and the generalized friction law $\xi \mu_b$ (up to order 1). It is thus possible to treat the remaining term R as it was treated in the incompressible case by Deleage and Richard 2025. Equation (5.111) can be written

$$\frac{\partial \tilde{r} \tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\tilde{r} \tilde{U}^2 + \tilde{r}^3 \tilde{\psi} + \frac{\tilde{r}^2 \cos \theta}{2F^2} \right) = \frac{\tilde{r} \cos \theta}{\varepsilon F^2} (\tan \theta - \xi \mu_b) - R \frac{\cos \theta}{F^2} + O(\varepsilon). \quad (5.126)$$

As in the incompressible case, we split R into two terms. The first term is written in the left-hand side of Equation (5.126) as a flux, and the second term is written in the

5. *Modèle consistant d'écoulement granulaire compressible – 4. Derivation of the depth-averaged equations*

right-hand side and is expressed as a relaxation term. More precisely, after introducing

$$\lambda = \frac{847(4 - 3C)}{20(157C + 84)}, \quad (5.127)$$

we obtain, as in the incompressible case,

$$\frac{\partial \tilde{r} \tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\tilde{r} \tilde{U}^2 + \beta \tilde{r}^3 \tilde{\psi} + K \frac{\tilde{r}^2 \cos \theta}{2F^2} \right) = \frac{\cos \theta}{\varepsilon F^2} \left[\tilde{r} (\tan \theta - \xi \mu_b) + \frac{10\lambda}{\tilde{A}^2(B + D)} \left(\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{r}^2} \right) \right] + O(\varepsilon), \quad (5.128)$$

where

$$\beta = \frac{11}{16} + \frac{5\lambda}{44}, \quad \text{and} \quad K = 1 - \frac{3C}{4} + \frac{\lambda C}{7}. \quad (5.129)$$

Equation (5.128) is consistent at the first order of accuracy. The first-order correction to the usual granular friction law is written partly with the corrective factor ξ , partly as a relaxation term in the right-hand side and partly in the momentum flux as corrections to the hydrostatic pressure term (K) and to the enstrophy term (β). The latter term also models shearing effects in the granular flow. The structure of this equation is similar to other models with enstrophy (Denisenko, Richard, et al. 2023 for example) with an effective enstrophy equal to $\beta\psi$. There is also a factor K in the hydrostatic pressure term. Such a factor has already been introduced to model normal stresses anisotropy (see for instance Savage et al. 1989). In the present case, the factor K enables to guarantee consistency up to order 1 with the rheology and is not directly linked with normal stresses anisotropy.

4.3. Depth-averaged energy equation

In order to close the system of equations obtained for $\tilde{h}$ and $\tilde{U}$ in the previous section, we need to find an evolution equation for the variable $\tilde{\psi}$. Such an equation can be obtained from the equation of energy (5.37). We first rewrite equation (5.37) as:

$$\frac{\partial \tilde{E}}{\partial \tilde{t}} + \text{div}(\tilde{E} \tilde{\mathbf{v}}) = \frac{1}{\varepsilon F^2} \tilde{u} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} - \frac{\tilde{\mathbf{v}} \cdot \nabla \tilde{p}}{F^2} + O(\varepsilon). \quad (5.130)$$

The integration of Equation (5.130) between 0 and $\tilde{h}$, using the boundary condition (5.38) gives

$$\frac{\partial}{\partial \tilde{t}} \left(\int_0^{\tilde{h}} \tilde{E} d\tilde{z} \right) + \frac{\partial}{\partial \tilde{x}} \left(\int_0^{\tilde{h}} \tilde{E} \tilde{u} d\tilde{z} \right) + \int_0^{\tilde{h}} \frac{\tilde{\mathbf{v}} \cdot \nabla \tilde{p}}{F^2} = \frac{1}{\varepsilon F^2} \int_0^{\tilde{h}} \tilde{u} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} d\tilde{z} + O(\varepsilon). \quad (5.131)$$

We compute that

$$\frac{1}{\phi^{(0)}} \int_0^{\tilde{h}} \tilde{E} d\tilde{z} = \frac{\tilde{r} \tilde{U}^2}{2} + \frac{\tilde{r}^3 \tilde{\psi}}{2} - \frac{\tilde{r} \tilde{x} \sin \theta}{\varepsilon F^2} + \frac{\tilde{r}^2}{2F^2} + O(\varepsilon). \quad (5.132)$$

5. *Modèle consistant d'écoulement granulaire compressible – 4. Derivation of the depth-averaged equations*

Similarly,

$$\frac{1}{\phi^{(0)}} \int_0^{\tilde{h}} \tilde{E} \tilde{u} d\tilde{z} = \frac{\tilde{r} \tilde{U}^3}{2} + \frac{3\tilde{r}^3 \tilde{\psi}}{2} + \frac{\tilde{r} \langle (\tilde{u} - \tilde{U})^3 \rangle}{2} - \frac{\tilde{r} \tilde{U} \tilde{x} \sin \theta}{\varepsilon F^2} + \int_0^{\tilde{h}} \frac{\phi \tilde{u} \tilde{z} \cos \theta}{\phi^{(0)} F^2} d\tilde{z}. \quad (5.133)$$

In order to close the equations, we need a formula for the third-order correlation $\langle (\tilde{u} - \tilde{U})^3 \rangle$. Since we neglect second-order terms, it is consistent to replace this term by the leading term in the asymptotic expansion, which is $\tilde{r} \langle (\tilde{u} - \tilde{U})^3 \rangle^{(0)} = -4\tilde{A}^3 \tilde{h}^{11/2}/1375$. Hence

$$\frac{\partial}{\partial \tilde{x}} \int_0^{\tilde{h}} \frac{1}{2} (\tilde{u}^{(0)} - \tilde{U}^{(0)})^3 d\tilde{z} = -\frac{\tilde{A}^3 \tilde{h}^{9/2}}{125} \frac{\partial \tilde{h}}{\partial \tilde{x}}. \quad (5.134)$$

This term is written in the right-hand side of the equation in order to obtain a model with a well-posed and simple mathematical structure. We now use the fact that the flow is incompressible at order 0, that $\tilde{p}(\tilde{h}) = O(\varepsilon)$ and the non-penetration condition to write that

$$\int_0^{\tilde{h}} \frac{\tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{p}}}{F^2} = \frac{\partial}{\partial \tilde{x}} \left(\int_0^{\tilde{h}} \frac{\tilde{u} \tilde{p}}{F^2} d\tilde{z} \right) + O(\varepsilon). \quad (5.135)$$

Equation (5.53) shows that, at order 0, we can write

$$\tilde{p}^{(0)} + \phi^{(0)} \tilde{z} \cos \theta = \phi^{(0)} \tilde{h} \cos \theta. \quad (5.136)$$

We can use this relation to write

$$\int_0^{\tilde{h}} \frac{\tilde{u}(\tilde{p} + \tilde{z} \cos \theta)}{\phi^{(0)} F^2} d\tilde{z} = \frac{\tilde{r}^2 \tilde{U} \cos \theta}{F^2} + O(\varepsilon). \quad (5.137)$$

We can also use the mass conservation (5.108) to let appear the total work of the gravity force as

$$\frac{\partial}{\partial \tilde{t}} \left(-\frac{\tilde{r} \tilde{x} \sin \theta}{\varepsilon F^2} \right) + \frac{\partial}{\partial \tilde{x}} \left(-\frac{\tilde{r} \tilde{U} \tilde{x} \sin \theta}{\varepsilon F^2} \right) = -\frac{\tilde{r} \tilde{U} \sin \theta}{\varepsilon F^2} \quad (5.138)$$

and put this term into the right-hand side of the depth-averaged energy equation. The technical details of the treatment of the right-hand side of Equation (5.131) are given in Appendix 7.2. The calculations are carried out to ensure compatibility between the energy and momentum depth-averaged equations in order to obtain a well-posed mathematical structure well suited to numerical resolution.

The final version of the energy equation can be written

$$\begin{aligned} & \frac{\partial}{\partial \tilde{t}} \left(\tilde{r} \frac{\tilde{U}^2}{2} + \frac{\beta \tilde{r}^3 \tilde{\psi}}{2} + K \frac{\tilde{r}^2 \cos \theta}{2F^2} \right) + \frac{\partial}{\partial \tilde{x}} \left(\tilde{r} \frac{\tilde{U}^3}{2} + \frac{3\beta \tilde{r}^3 \tilde{U} \tilde{\psi}}{2} + K \frac{\tilde{r}^2 \tilde{U} \cos \theta}{F^2} \right) \\ &= \frac{\tilde{U} \cos \theta}{\varepsilon F^2} \left[\tilde{r} (\tan \theta - \xi \mu_b) + \beta \frac{\tilde{\alpha}_1}{\tilde{r}^{1/2}} \left(\tilde{U} - \frac{2}{5} \tilde{A} \tilde{r}^{3/2} \right) + \left(\beta \tilde{\alpha}_2 + \frac{10\lambda}{\tilde{A}^2 (B + D)} \right) \left(\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{r}^2} \right) \right] + O(\varepsilon) \end{aligned} \quad (5.139)$$

with

$$\tilde{\alpha}_1 = -\frac{33C}{2\tilde{A}(B+D)(34C+28)}, \quad (5.140)$$

and

$$\tilde{\alpha}_2 = \frac{77(9C-12)}{2\tilde{A}^2(B+D)(34C+28)}. \quad (5.141)$$

As in the momentum equation, the effective enstrophy in this energy equation is $\beta\psi$ and there is the same factor K in the pressure term and in the potential energy. The right-hand side of this energy equation includes the power of the weight and of the usual granular friction law (5.114) but there are a corrective factor ξ and additional relaxation terms to obtain consistency at order 1.

The evolution equation for the variable $\tilde{\psi}$ can now be derived. In order to do this, the equation of momentum (5.128) is multiplied by $\tilde{U}$ and this new equation is subtracted from the equation of energy (5.139). This leads to

$$\frac{\partial \tilde{r}\tilde{\psi}}{\partial \tilde{t}} + \frac{\partial \tilde{r}\tilde{U}\tilde{\psi}}{\partial \tilde{x}} = \frac{2\tilde{U}\cos\theta}{\varepsilon\tilde{r}^2F^2} \left[\frac{\tilde{\alpha}_1}{\tilde{r}^{1/2}} \left(\tilde{U} - \frac{2}{5}\tilde{A}\tilde{r}^{3/2} \right) + \tilde{\alpha}_2 \left(\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{r}^2} \right) \right] + O(\varepsilon). \quad (5.142)$$

The evolution equation of the enstrophy is a transport equation with relaxation source terms. The relaxation term on ψ is important and usual for an enstrophy equation and the coefficient $\tilde{\alpha}_2$ must be negative for the system to be mathematically well-posed (otherwise the enstrophy diverges exponentially). This gives the condition $C < 4/3$, which is similar to the one obtained by Deleage and Richard 2025. When compressibility is taken into account in the definition of C , the condition $C < 4/3$ is satisfied for a wider range of slopes. In some sense, the system obtained from the compressible rheology is "more regular" than the one obtained from the incompressible rheology. See also §5.4.

5. Model analysis

5.1. Hyperbolicity

In this section, we study some properties of the model obtained in Section 4 and formed by the mass conservation equation (5.107), the momentum balance equation (5.128) and the enstrophy equation (5.142). In dimensional form, this system reads

$$\begin{cases} \frac{\partial r}{\partial t} + \frac{\partial rU}{\partial x} = 0, \\ \frac{\partial rU}{\partial t} + \frac{\partial}{\partial x} \left(rU^2 + \beta r^3\psi + K \frac{gr^2}{2} \cos\theta \right) = g \cos\theta \left[r(\tan\theta - \xi\mu_b) + \frac{10\lambda}{A^2(B+D)} \left(\psi - \frac{U^2}{4r^2} \right) \right], \\ \frac{\partial r\psi}{\partial t} + \frac{\partial rU\psi}{\partial x} = \frac{2gU\cos\theta}{r^2} \left[\frac{\alpha_1}{r^{1/2}} \left(U - \frac{2}{5}Ar^{3/2} \right) + \alpha_2 \left(\psi - \frac{U^2}{4r^2} \right) \right], \end{cases} \quad (5.143)$$

5. Modèle consistant d'écoulement granulaire compressible – 5. Model analysis

where the expressions of A , B , C , D , α_1 , α_2 and $\phi^{(0)}$ are

$$A = I_m \frac{\tan \theta - \mu_1}{\mu_2 - \tan \theta} \frac{\sqrt{\phi^{(0)} g \cos \theta}}{d}, \quad B = \frac{\mu_2 - \mu_1}{(\mu_2 - \tan \theta)(\tan \theta - \mu_1)}, \quad C = \tan \theta \left(2 \tan \theta - \frac{1}{B + D} \right), \quad (5.144)$$

$$D = -\frac{\partial \ln \phi}{\partial \mu} (\mu = \tan \theta), \quad \alpha_1 = -\frac{33C}{2A(B + D)(34C + 28)}, \quad \alpha_2 = \frac{77(9C - 12)}{2A^2(B + D)(34C + 28)}, \quad (5.145)$$

and

$$\phi^{(0)} = \phi \left(I = I_m \frac{\tan \theta - \mu_1}{\mu_2 - \tan \theta} \right). \quad (5.146)$$

The parameters λ , β and K are given by

$$\lambda = \frac{847(4 - 3C)}{20(157C + 84)}, \quad \beta = \frac{11}{16} + \frac{5\lambda}{44}, \quad K = 1 - \frac{3C}{4} + \frac{\lambda C}{7}, \quad (5.147)$$

and the generalized basal friction $\xi \mu_b$ is given by

$$\xi = \exp \left[-\frac{5D}{2Ar^{3/2}B(B + D)\tan \theta} \left(U - \frac{2}{5}Ar^{3/2} \right) \right], \quad (5.148)$$

and

$$\mu_b = \mu_1 + \frac{\mu_2 - \mu_1}{\frac{2r^{3/2}\sqrt{\phi^{(0)}g\cos\theta}}{I_m \frac{5dU}{5dU}} + 1}. \quad (5.149)$$

The three characteristic velocities are

$$\lambda_0 = U, \quad \lambda_{\pm} = U \pm \sqrt{Kgr\cos\theta + 3\beta r^2\psi}. \quad (5.150)$$

The system of equations (5.143) is hyperbolic if $K > 0$. It is in conservative form, with relaxation source terms. For the final expression (5.127) of λ or for $\lambda = 0$, the condition $K > 0$ is equivalent to the condition $C < 4/3$, which is also a condition for a well-posed system (see above in §4.3) and which is discussed in §5.4. If $K < 0$, the hyperbolicity is not guaranteed, which is not admissible.

The system admits the additional energy balance equation

$$\begin{aligned} & \frac{\partial re}{\partial t} + \frac{\partial}{\partial x} (rUe + \Pi U) \\ &= gU \cos \theta \left[r(\tan \theta - \xi \mu_b) + \frac{\beta \alpha_1}{r^{1/2}} \left(U - \frac{2}{5}Ar^{3/2} \right) + \left(\beta \alpha_2 + \frac{10\lambda}{A^2(B + D)} \right) \left(\psi - \frac{U^2}{4r^2} \right) \right] \end{aligned} \quad (5.151)$$

where the specific energy is

$$e = \frac{U^2}{2} + \frac{\beta r^2 \psi}{2} + K \frac{gr}{2} \cos \theta \quad (5.152)$$

and

$$\Pi = \beta r^3 \psi + K \frac{g r^2}{2} \cos \theta. \quad (5.153)$$

The field associated to λ_0 is linearly degenerate, hence the corresponding discontinuities are called contact discontinuities. The discontinuities associated to the eigenvalues $\lambda_{\pm}$ are shocks. Across a shock of speed s , the quantities $q = r(U - s)$ (conservation of mass) and ψ (conservation of enstrophy) are continuous. Furthermore, the jump of height h is constrained by the shock relation

$$q^2 \left[\frac{1}{r} \right] + \beta \psi [r^3] + \frac{K g \cos \theta}{2} [r^2] = 0, \quad (5.154)$$

where the notation $[f]$ stands for the jump of the quantity f .

It is important to note that the third equation of the system is the enstrophy equation and not the energy equation. This implies that, through a discontinuity, the enstrophy is conserved and the energy is dissipated. In the framework of hyperbolic systems of conservation laws, the energy is the mathematical entropy of the system.

5.2. Formula for the height and reconstruction of the velocity field

System (5.143) is written in terms of the three variables (r, U, ψ) . However, for comparison with experiments, it is useful to know the free surface h . We state here a formula which expresses h in terms of the variables (r, U, ψ) with the same accuracy as the model. To do so, we first express the height as

$$\tilde{h} = \tilde{r}^{(0)}. \quad (5.155)$$

Since $\tilde{r}^{(0)} = \tilde{r} - \varepsilon \tilde{r}^{(1)} + O(\varepsilon^2)$, it follows that $\tilde{h} = \tilde{r} - \varepsilon \tilde{r}^{(1)} + O(\varepsilon^2)$. From the equations (5.84), (5.96) and (5.106), we can express $\tilde{r}^{(1)}$ as

$$\tilde{r}^{(1)} = -\frac{10D}{\tilde{A}(B+D)} \left[\frac{\gamma_1}{\tilde{h}^{1/2}} \left(\tilde{U}^{(1)} - \frac{3}{5} \tilde{A} \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) + \frac{\gamma_2}{\tilde{A}} \left(\psi^{(1)} - \frac{\tilde{U}^{(0)} \tilde{U}^{(1)}}{2(\tilde{r}^{(0)})^2} + \frac{\tilde{r}^{(1)} (\tilde{U}^{(0)})^2}{2(\tilde{r}^{(0)})^3} \right) \right] \quad (5.156)$$

with

$$\gamma_1 = -\frac{7(15 \tan^2 \theta - 11C - 5)}{10(17C + 14)}, \quad \gamma_2 = \frac{231(25 \tan^2 \theta - 7C + 1)}{10(17C + 14)}. \quad (5.157)$$

With equations (5.95) and (5.105), it follows that, up to second order terms,

$$\tilde{h} = \tilde{r} \exp \left\{ \frac{10D}{\tilde{r} \tilde{A}(B+D)} \left[\frac{\gamma_1}{\tilde{r}^{1/2}} \left(\tilde{U} - \frac{2}{5} \tilde{A} \tilde{r}^{3/2} \right) + \frac{\gamma_2}{\tilde{A}} \left(\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{r}^2} \right) \right] \right\}. \quad (5.158)$$

Reverting to dimensional quantities, we get

$$h = r \exp \left\{ \frac{10D}{rA(B+D)} \left[\frac{\gamma_1}{r^{1/2}} \left(U - \frac{2}{5} Ar^{3/2} \right) + \frac{\gamma_2}{A} \left(\psi - \frac{U^2}{4r^2} \right) \right] \right\}. \quad (5.159)$$

As for the corrective factor ξ , the exponential function appearing in the formula of the reconstructed h can be substituted by any function f such that $f(0) = f'(0) = 1$. The choice $f = \exp$ guarantees that the reconstructed height is positive, for any values of (r, U, ψ) . Another possible choice is

$$f(x) = \frac{1}{\phi_c} \left[\phi^{(0)} + (\phi_c - \phi^{(0)}) \exp \left(\frac{\phi_c}{\phi_c - \phi^{(0)}} x \right) \right]. \quad (5.160)$$

This last expression yields an averaged volume fraction $\bar{\phi} = \phi^{(0)} r / h$ which stays between 0 and ϕ_c (see (5.82)). Close to stationary uniform flows, the two expressions are equivalent. A further study is necessary in order to obtain a reconstruction which stays valid far from steady uniform flows.

Since the model is derived consistently from the asymptotic expansions of the flow variables, it is also possible to reconstruct the 2D-fields and, in particular, the variations of the velocity profiles in the depth, from the values of the depth-averaged quantities calculated with the 1D-depth-averaged model. These profiles are calculated at the same accuracy as the model. In the following, we show how to reconstruct in a consistent way the horizontal velocity field from the solution of (5.143), i.e. from the triplet (r, U, ψ) .

Reverting to the dimensionless quantities, the expressions of $\tilde{u}^{(0)}$ at equilibrium (5.55) and of the averaged velocity $\tilde{U}^{(0)}$ (5.90) lead to

$$\tilde{u}^{(0)} = f(z) \tilde{U}^{(0)}, \quad (5.161)$$

where

$$f(z) = \frac{5}{3} \left[1 - \left(1 - \frac{z}{h} \right)^{3/2} \right] \quad (5.162)$$

is the normalised Bagnold profile. From this relation, we deduce that an approximation of $\tilde{u}$ at order 0 is given by $\tilde{u} = f(z) \tilde{U} + O(\varepsilon)$. In order to have an approximation of order 1, we expand

$$\tilde{u} = f(z) \tilde{U} + \varepsilon (\tilde{u}^{(1)} - f(z) \tilde{U}^{(1)}) + O(\varepsilon^2). \quad (5.163)$$

Thus we can obtain a consistent expression for $\tilde{u}$ by expressing consistently $\tilde{u}^{(1)} - f(z) \tilde{U}^{(1)}$ as a function of $(\tilde{h}, \tilde{U}, \tilde{\psi})$. After using (5.80) and (5.91) and some computations, we obtain

$$\tilde{u}^{(1)} - f(z) \tilde{U}^{(1)} = \tilde{A} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\tilde{h}^{3/2} g_1(z) + \tilde{h}^{7/2} \frac{F^2 \tilde{A}^2}{\cos \theta} g_2(z) \right], \quad (5.164)$$

where

$$g_1(z) = \frac{z}{h} \left(2B - 2D - \frac{1}{\tan \theta} \right) \tan^2 \theta - f(z) \tan^2 \theta \left(B - \frac{7D}{5} - \frac{1}{2 \tan \theta} \right) \quad (5.165)$$

and

$$g_2(z) = \left(\frac{B}{6} - \frac{D}{150} \right) \frac{f(z)}{5} - \frac{2B - 2D/5}{45} \left[1 - \left(1 - \frac{z}{h} \right)^3 \right]. \quad (5.166)$$

This yields, using (5.96) and (5.106),

$$\begin{aligned} \tilde{u}^{(1)} - f(z) \tilde{U}^{(1)} &= \frac{10}{B+D} f_1(z) \left(\tilde{U}^{(1)} - \frac{3}{5} \tilde{A} \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) \\ &\quad + \frac{10}{\tilde{A}(B+D)} f_2(z) \left(\psi^{(1)} - \frac{\tilde{U}^{(0)} \tilde{U}^{(1)}}{2(\tilde{r}^{(0)})^2} + \frac{\tilde{r}^{(1)} (\tilde{U}^{(0)})^2}{2(\tilde{r}^{(0)})^3} \right), \end{aligned} \quad (5.167)$$

where

$$f_1(z) = \frac{55C g_2(z) - 7g_1(z)}{34C + 28}, \quad (5.168)$$

and

$$f_2(z) = -385 \tilde{h}^{1/2} \frac{(3C - 4) g_2(z) - g_1(z)}{34C + 28}. \quad (5.169)$$

As we did before, we finally use (5.99) and (5.105) to replace the terms of first order in the right-hand side of (5.167) and, together with (5.163), we obtain, reverting to the dimensional quantities,

$$u = f(z) U + \frac{10}{B+D} f_1(z) \left(U - \frac{2A}{5} r^{3/2} \right) + \frac{10}{A(B+D)} f_2(z) \left(\psi - \frac{U^2}{4r^2} \right). \quad (5.170)$$

This expression gives the velocity profile in the flow to the accuracy of the model, i.e. to within terms of order 2.

5.3. Linear stability and second order extension

We study here the linear stability of a steady uniform solution of (5.143). In other words, we suppose that the solutions are sinusoidal perturbations of the constant solution given by $\tilde{r} = 1$, $\tilde{U} = \tilde{U}^{(0)}$ and $\tilde{\psi} = \tilde{\psi}^{(0)}$. The depth-averaged velocity can be scaled with its value in the constant solution, which yields $\tilde{A} = 5/2$, $\tilde{U}^{(0)} = 1$ and $\tilde{\psi}^{(0)} = 1/4$. We thus write $\tilde{r} = 1 + r'$, $\tilde{U} = 1 + U'$, $\tilde{\psi} = 1/4 + \psi'$, where $(r', U', \psi') = (a_1, a_2, a_3) \exp[ik(\tilde{x} - c\tilde{t})]$. The amplitudes a_1, a_2, a_3 are three constants, as well as the wave number k and the phase velocity c . We obtain the following linearised system:

$$M(\varepsilon) (r', U', \psi')^T = 0, \quad \text{with} \quad M(\varepsilon) = \frac{\cos \theta}{F^2} R + i\varepsilon k L \quad (5.171)$$

where the matrix L comes from the conservative terms of the left-hand side of the equations, and is given by

$$L = \begin{pmatrix} 1-c & 1 & 0 \\ \frac{3\beta}{4} + \frac{K \cos \theta}{F^2} & 1-c & \beta \\ 0 & 0 & 1-c \end{pmatrix}, \quad (5.172)$$

and the matrix R comes from the right-hand side relaxation terms and is given by

$$R = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{3}{2(B+D)} - \frac{4\lambda}{5(B+D)} & \frac{1}{B+D} + \frac{4\lambda}{5(B+D)} & -\frac{8\lambda}{5(B+D)} \\ 3\tilde{\alpha}_1 - \tilde{\alpha}_2 & -2\tilde{\alpha}_1 + \tilde{\alpha}_2 & -2\tilde{\alpha}_2 \end{pmatrix}, \quad (5.173)$$

The dispersion relation is then obtained by imposing the condition $\det(M(\varepsilon)) = 0$. It is exactly the same as in the incompressible case, except that the constant B of the incompressible case should be replaced by $B + D$ here. Hence the study of Deleage and Richard 2025 can be straightforwardly applied to the present model. We deduce that the condition of stability is $25F^2/4 < (4 - 3C) \cos \theta$. A consistent expression of the stability criterion for the incompressible $\mu(I)$ -rheology was obtained by Forterre 2006, using a slightly different Froude number $\mathcal{F}$, defined as $\mathcal{F}^2 = F^2 / \cos \theta$. With this definition of the Froude number, the obtained stability criterion coincides with the one found by Forterre 2006 in the incompressible case (i.e. when $D = 0$) and it reads

$$\mathcal{F} < \mathcal{F}_c = \frac{4}{5} \sqrt{1 - \frac{3C}{4}} = \frac{4}{5} \sqrt{1 - \frac{3}{2} \tan^2 \theta + \frac{3}{4} \frac{(\mu_2 - \tan \theta)(\tan \theta - \mu_1) \tan \theta}{\mu_2 - \mu_1 + D(\mu_2 - \tan \theta)(\tan \theta - \mu_1)}}. \quad (5.174)$$

The stability criteria are identical in the incompressible case because the present depth-averaged model is consistent up to order 1 in ε with the compressible $\mu(I)$ -rheology, which reduces to the incompressible rheology when $\text{div}(\mathbf{v}) = 0$. The stability criteria obtained for the compressible model is thus an extension of the formula found by Forterre 2006. In particular, this formula is also valid for the compressible equations of the $\mu(I) - \phi(I)$ rheology.

Note that the critical Froude number $\mathcal{F}_c$ is real if the condition $C < 4/3$ is satisfied. This condition is also a condition for a well-posed enstrophy equation (see §4.3 above) and to guarantee the hyperbolicity of the system (see §5.1). Since this condition appears in the stability criterion of a steady uniform flow for the $\mu(I)$ rheology, it is not specific to the model derived in this work. This condition is discussed in §5.4.

The variation of the consistent critical Froude number for the incompressible rheology (Forterre 2006) as a function of the inclination is presented in Figure 5.2 (red curve) in the case of glass beads, using the values $\mu_1 = \tan 20.9^\circ$ and $\mu_2 = \tan 32.76^\circ$ (Jop et al. 2005), together with the experimental measurements of Forterre and Pouliquen 2003 (black dots). The same curve corresponding to the compressible rheology is plotted in green. Since there is no data for the law $\phi(I)$ in the paper by Forterre and Pouliquen 2003, we chose for $\phi(I)$ the classical affine law (5.12), with standard parameters

$\phi_c = 0.6$, $\phi_m = 0.4$. The green curve is lower than the red one, although the two curves are very close to each other. The stability criterion obtained with a Saint-Venant model (Forterre and Pouliquen 2003) is also shown, with a shape factor of 1 (velocity constant over the depth) (black line), or with a shape factor of 5/4 (Bagnold velocity profile) (blue line).

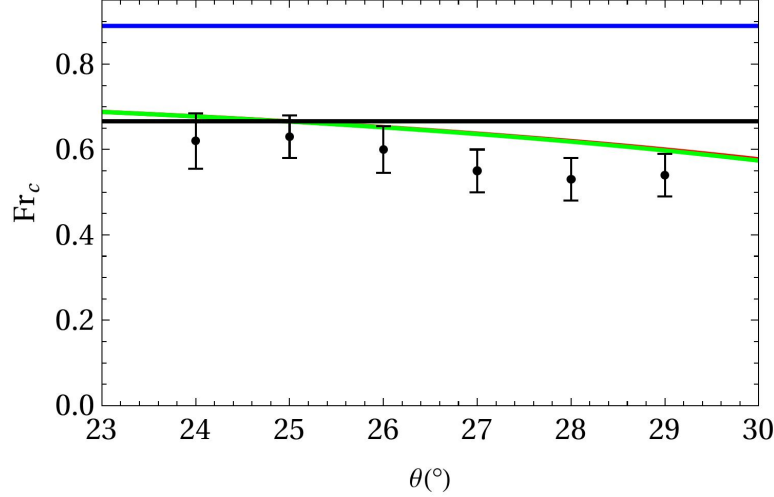


Figure 5.2. – Critical Froude number as a function of the angle θ for glass beads. Theoretical value obtained with the incompressible $\mu(I)$ rheology (Forterre 2006) (red curve) and given by the present model (green curve). Saint-Venant model with a shape factor of 1 (velocity constant over the depth) (black line) and of 5/4 (Bagnold velocity profile) (Forterre and Pouliquen 2003). Black dots: experimental measurements of Forterre and Pouliquen 2003.

As in the incompressible case, it is possible to write a higher order version of the model with diffusive terms, in order to capture the cut-off frequency which was observed by Forterre and Pouliquen 2003. The diffusive terms can be obtained from the expression of $\tilde{\tau}_{xx}^{(0)}$, see (5.58). Since the flow is incompressible at order 0, the computations of Deleage and Richard 2025 stay valid for the compressible case. We obtain the following system:

$$\begin{aligned}
 \frac{\partial r}{\partial t} + \frac{\partial rU}{\partial x} &= 0, \\
 \frac{\partial rU}{\partial t} + \frac{\partial}{\partial x} (rU^2 + \Pi) &= g \cos \theta \left[r(\tan \theta - \xi \mu_b) + \frac{10\lambda}{A^2 B} \left(\psi - \frac{U^2}{4r^2} \right) \right] \\
 &\quad + \frac{\partial}{\partial x} \left(\alpha_v r^{3/2} \frac{\partial U}{\partial x} \right), \\
 \frac{\partial r\psi}{\partial t} + \frac{\partial rU\psi}{\partial x} &= \frac{2gU \cos \theta}{r^2} \left[\frac{\alpha_1}{r^{1/2}} \left(U - \frac{2}{5} Ar^{3/2} \right) + \alpha_2 \left(\psi - \frac{U^2}{4r^2} \right) \right] \\
 &\quad + \frac{\partial}{\partial x} \left(\frac{16\alpha_v}{7\beta} r^{3/2} \frac{\partial \psi}{\partial x} \right),
 \end{aligned} \tag{5.175}$$

where

$$\alpha_v = \frac{5g \sin \theta}{9A}, \quad \Pi = \beta r^3 \psi + K \frac{gh^2}{2} \cos \theta, \quad (5.176)$$

and the other parameters are defined in §5.1. The system is compatible with the energy equation

$$\begin{aligned} & \frac{\partial r e}{\partial t} + \frac{\partial}{\partial x} (r U e + \Pi U) \\ &= g U \cos \theta \left[r (\tan \theta - \xi \mu_b) + \frac{\beta \alpha_1}{r^{1/2}} \left(U - \frac{2}{5} A r^{3/2} \right) + \left(\beta \alpha_2 + \frac{10\lambda}{A^2(B+D)} \right) \left(\psi - \frac{U^2}{4r^2} \right) \right] \\ &+ \frac{\partial}{\partial x} \left(\alpha_v r^{3/2} U \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial x} \left(\frac{8\alpha_v}{7\beta} r^{7/2} \frac{\partial \psi}{\partial x} \right) - \alpha_v r^{3/2} \left(\frac{\partial U}{\partial x} \right)^2 - \frac{16\alpha_v}{7\beta} r^{5/2} \frac{\partial r}{\partial x} \frac{\partial \psi}{\partial x}, \end{aligned} \quad (5.177)$$

with e defined by (5.152).

We can then extend the linear stability analysis to the study of the spatial growth rate, of the phase velocity, and of the cut-off frequency, as it was performed by Deleage and Richard 2025. Note that the only difference between the dispersion relation of the compressible model and the one of the incompressible model is that the parameter B is replaced by $B + D$ in the compressible case. Thus we do not expect a radical change in the linear properties of the two systems. We compare the predictions of the models with the experimental data of Forterre and Pouliquen 2003, in the case of glass beads. Hence we take the same parameters that were used in this paper: $\theta = 29^\circ$, $\mu_1 = \tan(20.9^\circ)$, $\mu_2 = \tan(32.76^\circ)$, $\mathcal{F} = 1.02$.

The variations of the dimensionless growth rate with the dimensionless pulsation are given in figure 5.3. The predictions of the compressible model (in red) are compared to the incompressible one (in blue). The dashed lines are for the purely hyperbolic systems, while the full curves are for the systems with diffusion. In both the incompressible and compressible cases, the addition of diffusion enables to recover a cut-off frequency, which is not predicted by the purely hyperbolic model. Taking compressibility into account seems to provide a system which is "less unstable", as the growth rate is a little smaller in the compressible case. In figure 5.4, we show the dimensionless phase velocities, using the same colour conventions. We see that all models give a similar phase velocity. Finally, the neutral stability curves are shown in figure 5.5. Again, the general shape of the curve stays similar when compressibility is added, even if the unstable zone is lightly restricted.

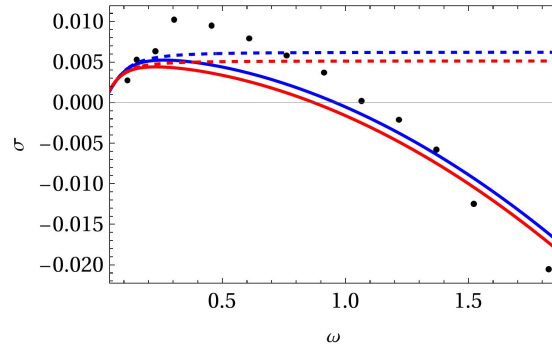


Figure 5.3. – Spatial dimensionless growth rate as a function of the dimensionless frequency. Black dots: experimental data of Forterre and Pouliquen 2003. Red curves: compressible models. Blue curves: incompressible models of Deleage and Richard 2025. Full lines: with diffusion. Dashed lines: without diffusion.

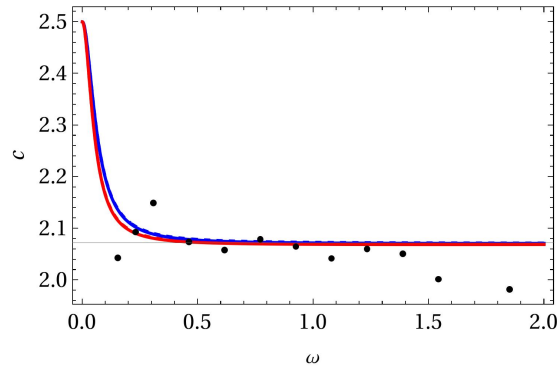


Figure 5.4. – Dimensionless phase velocity as a function of the dimensionless frequency. Black dots: experimental data of Forterre and Pouliquen 2003. Red curves: compressible models. Blue curves: incompressible models of Deleage and Richard 2025. Full lines: with diffusion. Dashed lines: without diffusion.

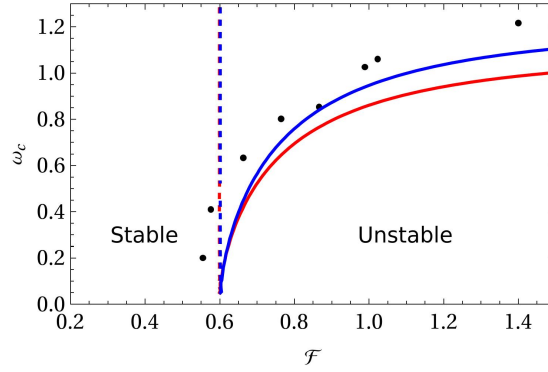


Figure 5.5. – Dimensionless cut-off frequency as a function of the Froude number. Black dots: experimental data of Forterre and Pouliquen 2003. Red curves: compressible models. Blue curves: incompressible models of Deleage and Richard 2025. Full lines: with diffusion. Dashed lines: without diffusion.

5.4. Domain of well-posedness

As in the incompressible case, the systems that are written here are only valid if the two following conditions hold:

$$\mu_1 < \tan \theta < \mu_2, \quad (5.178)$$

and

$$C < \frac{4}{3}. \quad (5.179)$$

These two conditions are inherited from the $\mu(I)$ -rheology. Indeed, condition (5.178) states that the computations performed here are only valid on the slopes at which stationary uniform flows are solutions of the constitutive equations of the compressible $\mu(I)$ -rheology. Similarly, if condition (5.179) is not satisfied, every stationary uniform solution of the compressible rheology constitutive equations is unstable (Forterre 2006; Deleage and Richard 2025). We refer to section 6 of chapter 3 for more comments on these two ill-posed regimes.

Note that it is clear that these ill-posedness issues are not linked with the Hadamard instability of the incompressible $\mu(I)$ -rheology studied by Barker, Schaeffer, Bohórquez, et al. 2015. Indeed, the systems are derived from a compressible version of the rheology, for which this instability disappears as soon as equation (5.15) is satisfied (see Heyman et al. 2017).

6. Conclusion

In this work, we presented a depth-averaged model for granular flow consistent with the compressible $\mu(I)$ rheology. The model is derived with the help of asymptotic

expansions of the variables up to order 1 in powers of the shallow water parameter ε . The compressibility is taken into account in the averaging procedure with the help of the Favre averaging operator. As a consequence, we obtain a three-equation model, for three variables: the mass hold-up, the Favre averaged horizontal velocity, and the enstrophy. The enstrophy is defined with the help of the second correlations of the velocity and enables to take into account shear effects. The three equations form a hyperbolic system of conservation laws with source terms, compatible with an energy balance equation. We show that the system obtained is similar to a previous system that was obtained by Deleage and Richard 2025 for an incompressible flow. Taking compressibility into account results in adding a corrective factor to the friction law, which slightly attenuates the rate at which friction increases with the mean velocity, and redefining a parameter of the final model.

The height of the flow and the vertical velocity profile can be reconstructed from the three variables of the system, with the same accuracy than the one of the model (i.e. with an error of $O(\varepsilon^2)$). The roll waves instability of the model is also studied. It is shown that it appears when the Froude number is above a critical threshold. Since the model is consistent up to order 1 with the $\mu(I) - \phi(I)$ rheology, the threshold is the same for the bulk equation in the long wave limit. In particular, we obtain for the first time the expression of the theoretical instability threshold of the $\mu(I) - \phi(I)$ rheology in the long wave limit. A higher order version of the model with diffusive terms similar to the ones of Gray and Edwards 2014; Deleage and Richard 2025 is also presented. We show that these terms enable to recover a cut-off frequency similar to the one measured by Forterre and Pouliquen 2003.

7. Appendix

7.1. Case of a non-vanishing bulk friction coefficient

We show here that adding a bulk-friction coefficient into the constitutive equations does not change the asymptotic expansions of the variables up to order 1 obtained in section 3. Indeed, we will see that it results in the addition of terms of order $O(\varepsilon^2)$ in the equations.

We thus consider a Cauchy stress tensor given by (5.14):

$$\boldsymbol{\sigma} = -p(\phi, |\mathbf{S}|)\mathbf{I}_d + \mu[I(\phi)]p(\phi, |\mathbf{S}|)\frac{\mathbf{S}}{|\mathbf{S}|} + 2\mu_v(\phi)p(\phi, |\mathbf{S}|)\frac{\text{div}(\mathbf{v})}{|\mathbf{S}|}\mathbf{I}_d, \quad (5.180)$$

with $p(\phi, |\mathbf{S}|)$ defined in (5.13). With the bulk friction coefficient, a diagonal matrix is added to the deviatoric stress tensor $\boldsymbol{\tau}$. It follows that only the diagonal coefficients of $\boldsymbol{\tau}$ are modified, and the coefficient τ_{xz} is unchanged. In dimensionless variables, we obtain that

$$\tilde{\tau}_{xx} = \mu(I)\frac{\tilde{p}}{|\tilde{\mathbf{S}}|}\left(\frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{w}}{\partial \tilde{z}}\right) + 2\mu_v(\phi)\frac{\tilde{p}}{|\tilde{\mathbf{S}}|}\left(\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}}\right), \quad (5.181)$$

and

$$\tilde{\tau}_{zz} = \mu(I) \frac{\tilde{p}}{|\tilde{\mathbf{S}}|} \left(\frac{\partial \tilde{w}}{\partial \tilde{z}} - \frac{\partial \tilde{u}}{\partial \tilde{x}} \right) + 2\mu_\nu(\phi) \frac{\tilde{p}}{|\tilde{\mathbf{S}}|} \left(\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} \right), \quad (5.182)$$

We see that these quantities appear with a factor at least ε in the momentum equations (5.35) and (5.36). Thus they do not influence these equations at order 0, and equations (5.41), (5.42), (5.43), (5.44) are still valid. Hence the system is again incompressible at order 0. Furthermore, their values at order 0, which appear in the equation of momentum of first order (see (5.59) and (5.60)), do not depend on the bulk friction coefficient μ_ν , which is multiplied by $\text{div}(\mathbf{v}) = 0$ at order 0. Hence all expressions for quantities at order 1 that were derived in §3.2 are unchanged, and the derivation presented in section 4 is still consistent at order 1 with the equations. Similarly, the derivation of the depth-averaged diffusion terms, which only uses the value of $\tilde{\tau}_{xx}^{(0)}$, is not affected by the bulk friction coefficient.

7.2. Derivation of the depth-averaged energy equation

We compute the right-hand-side of (5.131). In order to do this, we use the expansion (5.117) to keep track of the friction law (5.114). We first compute

$$\begin{aligned} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} &= \frac{\partial}{\partial \tilde{z}} (\tilde{\tau}_{xz}^{(0)} + \varepsilon \tilde{\tau}_{xz}^{(1)}) + O(\varepsilon^2) = -\mu^{(0)} \phi^{(0)} \cos \theta + \varepsilon \frac{\partial \tilde{\tau}_{xz}^{(1)}}{\partial \tilde{z}} + O(\varepsilon^2) \\ &= -\xi \mu_b \phi^{(0)} \cos \theta + \varepsilon \frac{\phi^{(0)} \cos \theta}{(B+D) \tilde{U}^{(0)}} \left(\tilde{U}^{(1)} - \frac{3}{5} \tilde{A} \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) + \varepsilon \frac{\partial \tilde{\tau}_{xz}^{(1)}}{\partial \tilde{z}} + O(\varepsilon^2), \end{aligned} \quad (5.183)$$

where the last equality comes from (5.122). We obtain with (5.55) and (5.78)

$$\begin{aligned} \int_0^{\tilde{h}} \tilde{u} \frac{\partial \tilde{\tau}_{xz}}{\partial \tilde{z}} d\tilde{z} &= \int_0^{\tilde{h}} \tilde{u} \phi \cos \theta \left[-\xi \mu_b + \varepsilon \frac{1}{(B+D) \tilde{U}^{(0)}} \left(\tilde{U}^{(1)} - \frac{3}{5} \tilde{A} \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) \right] d\tilde{z} \\ &\quad + \varepsilon \int_0^{\tilde{h}} \tilde{u}^{(0)} \left(\frac{\partial \tilde{\tau}_{xz}^{(1)}}{\partial \tilde{z}} + \phi^{(1)} \sin \theta \right) d\tilde{z} + O(\varepsilon^2) \end{aligned} \quad (5.184)$$

$$= -\phi^{(0)} \tilde{r} \tilde{U} \xi \mu_b \cos \theta + \varepsilon \frac{\phi^{(0)} \tilde{h} \cos \theta}{B+D} \left(\tilde{U}^{(1)} - \frac{3}{5} \tilde{A} \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) \quad (5.185)$$

$$+ \varepsilon \phi^{(0)} \tilde{h} \cos \theta \tilde{A} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{2}{5} \tilde{h}^{3/2} - \frac{F^2 \tilde{A}^2 \tilde{h}^{7/2}}{10 \cos \theta} \right) + O(\varepsilon^2). \quad (5.186)$$

We compute

$$\frac{1}{B+D} \left(\tilde{U}^{(1)} - \frac{3}{5} \tilde{A} \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) + \tilde{A} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left(\frac{2}{5} \tilde{h}^{3/2} - \frac{F^2 \tilde{A}^2}{10 \cos \theta} \tilde{h}^{7/2} \right) = \frac{\tilde{A}}{10} \frac{\partial \tilde{h}}{\partial \tilde{x}} 3C \tilde{h}^{3/2}. \quad (5.187)$$

We thus obtain the following equation for the energy:

$$\begin{aligned} & \frac{\partial}{\partial \tilde{t}} \left(\tilde{r} \frac{\tilde{U}^2}{2} + \frac{\tilde{r}^3 \tilde{\psi}}{2} + \frac{\tilde{r}^2 \cos \theta}{2F^2} \right) + \frac{\partial}{\partial \tilde{x}} \left(\frac{\tilde{r} \tilde{U}^3}{2} + \frac{3}{2} \tilde{r}^3 \tilde{U} \tilde{\psi} + \frac{\tilde{r}^2 \tilde{U}}{F^2} \cos \theta \right) \\ &= \frac{\tilde{r} \tilde{U} \cos \theta}{\varepsilon F^2} (\tan \theta - \xi \mu_b) + \frac{\tilde{A}}{10} \frac{\partial \tilde{h}}{\partial \tilde{x}} \left[\frac{2}{25} \tilde{A}^2 \tilde{h}^{9/2} + \tilde{h}^{5/2} \frac{\cos \theta}{F^2} 3C \right] + O(\varepsilon). \end{aligned} \quad (5.188)$$

To ensure compatibility between the energy equation and the momentum equation, we need to modify the left-hand side of (5.188). We compute with (5.90), (5.100) and (5.107)

$$\frac{1}{2} \frac{\partial \tilde{r}^3 \tilde{\psi}}{\partial \tilde{t}} + \frac{3}{2} \frac{\partial \tilde{r}^3 \tilde{U} \tilde{\psi}}{\partial \tilde{x}} = \frac{13}{250} \tilde{A}^3 \tilde{h}^{9/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} + O(\varepsilon). \quad (5.189)$$

Similarly,

$$\frac{\partial}{\partial \tilde{t}} \frac{\tilde{r}^2 \cos \theta}{2F^2} + \frac{\partial}{\partial \tilde{x}} \frac{\tilde{r}^2 \tilde{U} \cos \theta}{F^2} = \frac{2}{5} \tilde{A} \tilde{h}^{5/2} \frac{\cos \theta}{F^2} \frac{\partial \tilde{h}}{\partial \tilde{x}} + O(\varepsilon). \quad (5.190)$$

We now subtract Equations $(1 - \beta)$ (5.189) + $(1 - K)$ (5.190) from (5.188) to obtain

$$\begin{aligned} & \frac{\partial}{\partial \tilde{t}} \left(\tilde{r} \frac{\tilde{U}^2}{2} + \frac{\beta \tilde{r}^3 \tilde{\psi}}{2} + K \frac{\tilde{r}^2 \cos \theta}{2F^2} \right) + \frac{\partial}{\partial \tilde{x}} \left(\frac{\tilde{r} \tilde{U}^3}{2} + \frac{3\beta \tilde{r}^3 \tilde{U} \tilde{\psi}}{2} + K \frac{\tilde{r}^2 \tilde{U} \cos \theta}{F^2} \right) \\ &= \frac{\tilde{r} \tilde{U} \cos \theta}{\varepsilon F^2} (\tan \theta - \xi \mu_b) + \frac{\tilde{A}^3}{10} \tilde{h}^{9/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} \frac{13\beta - 11}{25} + \frac{2\lambda}{35} \tilde{A} C \tilde{h}^{5/2} \frac{\cos \theta}{F^2} \frac{\partial \tilde{h}}{\partial \tilde{x}} + O(\varepsilon). \end{aligned} \quad (5.191)$$

We recognize the work associated to the basal friction in the first term of the right-hand side. We still need to find a suitable expression for the remaining terms. First, it follows from the definition of β (5.129), the expression of $\tilde{U}^{(0)}$ (5.90) and Equation (5.106) that

$$\begin{aligned} & \frac{\tilde{A}^3}{10} \tilde{h}^{9/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} \frac{13\beta - 11}{25} + \frac{2\lambda \cos \theta}{35 F^2} \tilde{A} C \frac{\partial \tilde{h}}{\partial \tilde{x}} \tilde{h}^{5/2} \\ &= -\frac{3\beta}{250} \tilde{A}^3 \tilde{h}^{9/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} + \lambda \frac{10}{\tilde{A}^2 (B + D)} \frac{\cos \theta}{F^2} \tilde{U}^{(0)} \left(\psi^{(1)} - \frac{\tilde{U}^{(0)} \tilde{U}^{(1)}}{2(\tilde{r}^{(0)})^2} + \frac{\tilde{r}^{(1)} (\tilde{U}^{(0)})^2}{2(\tilde{r}^{(0)})^3} \right). \end{aligned} \quad (5.192)$$

By using equations (5.90), (5.96) and (5.106), we get the following exact equality:

$$\begin{aligned} -\frac{3}{250} \tilde{A}^3 \tilde{h}^{9/2} \frac{\partial \tilde{h}}{\partial \tilde{x}} &= \frac{\cos \theta}{F^2} \tilde{U}^{(0)} \left[\frac{\tilde{\alpha}_1(\theta)}{\tilde{h}^{1/2}} \left(\tilde{U}^{(1)} - \frac{3}{5} \tilde{A} \tilde{h}^{3/2} \frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} \right) \right. \\ &\quad \left. + \tilde{\alpha}_2(\theta) \left(\psi^{(1)} - \frac{\tilde{U}^{(0)} \tilde{U}^{(1)}}{2(\tilde{r}^{(0)})^2} + \frac{\tilde{r}^{(1)} (\tilde{U}^{(0)})^2}{2(\tilde{r}^{(0)})^3} \right) \right], \end{aligned} \quad (5.193)$$

where

$$\alpha_1(\theta) = -\frac{33C}{2A(B + D)(34C + 28)}, \quad \tilde{\alpha}_1(\theta) = -\frac{33C}{2\tilde{A}(B + D)(34C + 28)}, \quad (5.194)$$

and

$$\alpha_2(\theta) = \frac{77(9C - 12)}{2A^2(B + D)(34C + 28)}, \quad \tilde{\alpha}_2(\theta) = \frac{77(9C - 12)}{2\tilde{A}^2(B + D)(34C + 28)}. \quad (5.195)$$

Recall that $\tilde{U}^{(0)} = \tilde{U} + O(\varepsilon)$. Similarly, equation (5.89) can be written

$$\tilde{U}^{(1)} - \frac{3}{5}\tilde{A}\tilde{h}^{3/2}\frac{\tilde{r}^{(1)}}{\tilde{r}^{(0)}} = \frac{1}{\varepsilon} \left[\tilde{U} - \frac{2}{5}\tilde{A}\tilde{r}^{3/2} \right] + O(\varepsilon), \quad (5.196)$$

and similarly for (5.105)

$$\psi^{(1)} - \frac{\tilde{U}^{(0)}\tilde{U}^{(1)}}{2(\tilde{r}^{(0)})^2} + \frac{\tilde{r}^{(1)}(\tilde{U}^{(0)})^2}{2(\tilde{r}^{(0)})^3} = \frac{1}{\varepsilon} \left(\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{r}^2} \right) + O(\varepsilon). \quad (5.197)$$

We can thus modify (5.193) and obtain

$$-\frac{3}{250}\tilde{A}^3\tilde{h}^{9/2}\frac{\partial\tilde{h}}{\partial\tilde{x}} = \frac{\tilde{U}\cos\theta}{\varepsilon F^2} \left[\frac{\tilde{\alpha}_1(\theta)}{\tilde{r}^{1/2}} \left(\tilde{U} - \frac{2}{5}\tilde{A}\tilde{r}^{3/2} \right) + \tilde{\alpha}_2(\theta) \left(\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{r}^2} \right) \right] + O(\varepsilon). \quad (5.198)$$

In a similar fashion, we have

$$\lambda \frac{10}{\tilde{A}^2(B + D)} \frac{\cos\theta}{F^2} \tilde{U}^{(0)} \frac{\partial\tilde{h}}{\partial\tilde{x}} \left(\tilde{\psi}^{(1)} - \frac{\tilde{A}\tilde{U}^{(1)}}{5\tilde{h}^{1/2}} \right) = \lambda \tilde{U} \frac{10\cos\theta}{\varepsilon F^2 \tilde{A}^2(B + D)} \left(\tilde{\psi} - \frac{\tilde{U}^2}{4\tilde{r}^2} \right) + O(\varepsilon). \quad (5.199)$$

Troisième partie

Étude mathématique d'un modèle jouet de suspension

6. Stabilité de solutions de type onde progressive partiellement congestionnée pour un modèle jouet de suspension

Dans ce chapitre, on montre la stabilité non-linéaire d'une classe de solutions ondes progressives au système d'équations de Navier-Stokes compressible sans pression avec un coefficient de diffusion singulier. Ces équations sont inspirées d'une rhéologie de suspension granulaire et reproduisent l'augmentation de la résistance à la déformation du milieu avec l'augmentation de la fraction volumique solide. Grâce à l'introduction d'une "vitesse effective", ce système peut également être interprété comme un modèle social (système d'Aw-Rascle). La vitesse effective s'interprète alors comme la vitesse que les agents souhaitent atteindre, et dépend de la densité du milieu. Les solutions considérées ici encodent des effets de congestion en connectant un état congestionné à gauche à un état non congestionné à droite, et peuvent aussi être interprétées comme des solutions approchées d'un modèle de congestion dure. En utilisant des estimations d'énergie à poids, on prouve la stabilité non linéaire d'ondes de choc visqueux de ce système pour des perturbations d'intégrale nulle, ce qui étend en particulier des résultats précédents qui ne s'appliquent pas au cas où le coefficient de diffusion est singulier. Ce chapitre est extrait du preprint suivant, qui a été soumis pour publication (DELEAGE et MEHMOOD 2024) :

Deleage, E., & Mehmood, M. A. (2024). Stability of partially congested travelling wave solutions for the extended Aw-Rascle system. *arXiv preprint arXiv :2404.17406*.

1. Introduction

1.1. Presentation of the model

We study the following generalised Aw-Rascle system on the real line:

$$\begin{cases} \partial_t \rho + \partial_y(\rho u) = 0, \\ \partial_t(\rho w) + \partial_y(\rho u w) = 0. \end{cases} \quad (6.1)$$

Here, the quantity ρ represents the density while u and w respectively refer to the actual and desired velocities of agents. This system was originally coined in 2000 by

Aw and Rascle (Aw et al. 2000) and has popularly been used to model the evolution of a system of interacting agents, such as the flow of vehicular traffic (Sheng et al. 2022; Chaudhuri, Navoret, et al. 2024; Mehmood 2024) or crowd dynamics (Aceves-Sánchez et al. 2024). The standard Aw-Rascle system is complemented by the relation $w = u + P(\rho)$, where $P = P(\rho)$ is known as the ‘offset’ function. In this paper, we consider the case where $w = u + \partial_y p_\epsilon(\rho)$, with p_ϵ being a singular function of the density depending on a parameter ϵ . More precisely, for $\epsilon > 0$ fixed, we take

$$w = u + \partial_y p_\epsilon(\rho) = u + \partial_y \left(\epsilon \left(\frac{\rho}{1-\rho} \right)^\gamma \right), \quad \gamma \geq 1. \quad (6.2)$$

Note that the singularity as ρ approaches 1 in (6.2) is physically significant, since it implies that the density of agents within the system may not exceed a maximal packing constraint, i.e. $\rho \leq 1$. In one-dimension it is interesting to note that the system may be formally rewritten as

$$\begin{cases} \partial_t \rho + \partial_y(\rho u) = 0, \\ \partial_t(\rho u) + \partial_y(\rho u^2) - \partial_y(\rho^2 p'_\epsilon(\rho) \partial_y u) = 0, \end{cases} \quad (6.3)$$

which resembles the one-dimensional compressible pressureless Navier-Stokes equations with a singular degenerate viscosity coefficient $\lambda_\epsilon := \rho^2 p'_\epsilon(\rho)$.

The system (6.3) with (6.2) was rigorously derived by Lefebvre-Lepot and Maury in Lefebvre-Lepot et al. 2008 from a microscopic lubrication model, and describes the evolution of particles suspended in a viscous fluid that interact with each other via a lubricating force. The viscosity coefficient obtained by Lefebvre-Lepot and Maury is $\epsilon(1-\rho)^{-1}$, where ϵ is the viscosity of the interstitial fluid. The case $\gamma = 1$ has also proven to be physically relevant in applications. Indeed, the viscosity coefficient of dense granular suspensions was experimentally measured to behave like $(\phi_c - \phi)^{-2}$ as the solid volume fraction ϕ (which corresponds to the non-dimensionalised density) approaches the maximal volume fraction ϕ_c (see for instance Guazzelli et al. 2018 for a review).

Let us now go back to the momentum equation of (6.3). In the region where ρ is far from 1, the viscosity λ_ϵ vanishes uniformly as ϵ goes to 0 and (6.3) formally degenerates towards the pressureless gas system (Berthelin 2002; Boudin 2000). On the other hand, in the congested region, ρ is very close to 1, and the singularity $(1-\rho)^{-(\gamma+1)}$ compensates the degeneracy in ϵ . The limit $\epsilon \rightarrow 0$ in model (6.3)-(6.2) (see Chaudhuri, Navoret, et al. 2024) is then viewed as a transition towards a granular suspension model, where the interacting force is governed by the contact between the solid grains. The Aw-Rascle system with this choice of offset function was investigated by the authors of Chaudhuri, Navoret, et al. 2024 on a one-dimensional periodic domain, where the global well-posedness for fixed ϵ was studied, as well as the limit $\epsilon \rightarrow 0$, which is known as the ‘hard-congestion limit’. The authors demonstrated that up to a subsequence, solutions of (6.1)-(6.2) converge towards weak solutions of the

'hard-congestion model' (see also Mehmood [2024](#); Bresch, Perrin, et al. [2014](#)):

$$\begin{cases} \partial_t \rho + \partial_y(\rho u) = 0, & (6.4a) \\ \partial_t(\rho u + \partial_y \pi) + \partial_y((\rho u + \partial_y \pi)u) = 0, & (6.4b) \\ 0 \leq \rho \leq 1, (1 - \rho)\pi = 0, \pi \geq 0. & (6.4c) \end{cases}$$

In this sense, the original system (6.1)-(6.2) may be referred to as an approximation of (6.4). System (6.4) is an example of a free-congested system, where the free phase refers to the region where $\rho < 1$ and the congested region is where $\rho = 1$. The potential π is obtained in the limit and can be viewed as a Lagrange multiplier associated with the incompressibility constraint $\partial_x u = 0$ which holds in the congested region. The existence of strong solutions to the system (6.4) is still not known and there are also no results on the stability of non-trivial solutions (i.e. where π is not identically 0). Note however that the existence of weak and measure-valued (duality) solutions to this model was recently obtained on the real line in Chaudhuri, Mehmood, et al. [2024](#). We also refer to Chaudhuri, Gwiazda, et al. [2023](#); Garavello et al. [2011](#); Hu et al. [2020](#) and the references therein for further examples of the theoretical and numerical analysis pertaining to the Aw-Rascle system.

In the present work, we are concerned with the stability of a specific class of solutions to the system (6.1)-(6.2), namely the solutions (ρ, u) which are travelling waves that connect a congested left state ($\rho = 1$) to a non-congested right state ($\rho < 1$). Since systems (6.1)-(6.2) and (6.3) are equivalent (for sufficiently regular solutions), our task is closely related to the stability of travelling wave solutions to the compressible Navier-Stokes system, which has been studied by Dalibard and Perrin in Dalibard et al. [2019](#). There, the authors studied system (6.3) with the addition of a pressure and a constant viscosity coefficient $\mu > 0$. A similar study was also carried out in Vasseur et al. [2016](#) by Vasseur and Yao. Both of these works make use of a new formulation of the system, which is obtained by introducing the 'effective velocity' w and rewriting the system in terms of (w, v) , where $v := 1/\rho$ is the specific volume. The parabolic equation satisfied by v and the transport structure for w then allows for desirable energy estimates, which are carried out with the help of integrated variables (see Equation (6.12) below). Rewriting the system using the effective velocity has also proven to be advantageous when investigating the existence and uniqueness of weak/strong solutions to the compressible Navier-Stokes system with a density-dependent viscosity of the form $\mu(\rho) = \rho^\alpha$ for $\alpha > 0$ (see Burtea and Haspot [2020](#); Constantin et al. [2020](#) for example).

To the best of our knowledge, there are no known results concerning the stability of shock waves where the viscosity coefficient is singular. The interest in such a result is twofold. Firstly, the stability of strong solutions to (6.1)-(6.2) is significant due to the equivalence with the compressible Navier-Stokes system (6.3), for which such a result does not exist in the literature as far as we know. On the other hand, a stability result for partially congested solutions would imply that the system (6.4) is also expected to be stable, which provides additional validity for the model (6.4) and further verifies the need for a rigid well-posedness theory. Note that in our case the presence of a

singular, degenerate viscosity coefficient and the lack of pressure prevents us from using the arguments of Dalibard et al. 2019; Vasseur et al. 2016 to obtain the estimates required to prove the global existence and stability of travelling wave solutions to (6.1)-(6.2). Nonetheless, we demonstrate in this paper that through a careful choice of weighted energy estimates and the identification of good unknowns taking congestion into account, we can obtain results analogous to those of Dalibard et al. 2019 for the Aw-Rascle system (6.1) with the singular offset function (6.2).

Let us now give an outline of the paper. In the next subsection we detail our main results, which are the existence of travelling wave solutions to system (6.1)-(6.2), and the nonlinear stability of these solutions. Then in Section 2 we introduce basic properties of travelling wave solutions, an integrated formulation of the system and some useful preliminary estimates. The bulk of our work goes into Section 3, where we obtain the well-posedness of the integrated system. Lastly, we prove the equivalence between the integrated system and the original system in Section 4.

1.2. Main results

We now mention our main results. Let the Lagrangian mass coordinate x be such that $dx = \rho dy - \rho u dt$, and $v := 1/\rho$ the specific volume. Then (6.3) becomes

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u - \partial_x(\phi_\epsilon(v)\partial_x u) = 0, \end{cases} \quad (6.5)$$

where

$$\phi_\epsilon(v) := \frac{p'_\epsilon(1/v)}{v^3} = \frac{\epsilon\gamma}{v(v-1)^{\gamma+1}}, \quad \text{such that} \quad \partial_y p_\epsilon(\rho) = -\phi_\epsilon(v)\partial_x v. \quad (6.6)$$

In these coordinates, it follows that

$$w = u + \partial_y p_\epsilon(\rho) = u - \phi_\epsilon(v)\partial_x v$$

is constant, i.e. solves $\partial_t w = 0$.

Our first lemma establishes the existence of travelling wave solutions, and gives a quantitative description of the profile.

Proposition 6.1. *Let $1 = v_- < v_+$ and $u_- > u_+$ be real numbers. Then there exists a unique (up to a translation) travelling wave solution $(\mathbf{u}, \mathbf{v})(t, x) = (u_\epsilon, v_\epsilon)(\xi)$ of (6.5), complemented with the far field condition $(\mathbf{u}, \mathbf{v}) \rightarrow (u_\pm, v_\pm)$ as $\xi \rightarrow \pm\infty$, where $\xi := x - st$ and s is the shock speed which satisfies*

$$s = \frac{u_- - u_+}{v_+ - 1}. \quad (6.7)$$

The solution is a smooth monotone increasing function connecting 1 (at $-\infty$) to v_+ (at $+\infty$).

By setting $v_\epsilon(0) = (1 + v_+)/2$, one then has the following estimates:

6. Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet
de suspension – 1. Introduction

— In the congested region ($\xi < 0$),

$$1 + \left(B - \frac{A_0}{\epsilon} \xi\right)^{-1/\gamma} \leq v_\epsilon(\xi) \leq 1 + \left(B - \frac{A_1}{\epsilon} \xi\right)^{-1/\gamma}, \quad (6.8)$$

where

$$A_0 := \frac{s(v_+ - 1)(v_+ + 1)}{2}, \quad A_1 := \frac{s(v_+ - 1)}{2}, \quad \text{and} \quad B := \left(\frac{2}{v_+ - 1}\right)^\gamma.$$

— In the free region $\xi > 0$,

$$v_+ - \frac{v_+ - 1}{2} \exp\left(-\frac{A_2}{\epsilon} \xi\right) \leq v_\epsilon(\xi) \leq v_+ - \frac{v_+ - 1}{2} \exp\left(-\frac{A_3}{\epsilon} \xi\right), \quad (6.9)$$

where

$$A_2 := \frac{s(v_+ + 1)(v_+ - 1)^{\gamma+1}}{2\gamma^2\gamma} \quad \text{and} \quad A_3 := \frac{sv_+(v_+ - 1)^{\gamma+1}}{\gamma}.$$

It follows that v_ϵ converges almost everywhere to the shock wave $v(\xi) := \mathbf{1}_{\xi < 0} + v_+ \mathbf{1}_{\xi > 0}$ as $\epsilon \rightarrow 0$, and it holds that

$$v_\epsilon(\xi) \leq \left[1 + \left(B - \frac{A_1}{\epsilon} \xi\right)^{-1/\gamma}\right] \mathbf{1}_{\xi < 0} + v_+ \mathbf{1}_{\xi \geq 0}. \quad (6.10)$$

With this in hand, we dedicate the rest of our effort towards studying the stability of the profiles (u_ϵ, v_ϵ) where $\epsilon \ll 1$. We first express System (6.5) in term of the unknowns v and $w = u - \phi_\epsilon(v) \partial_x v$:

$$\begin{cases} \partial_t w = 0, \\ \partial_t v - \partial_x w - \partial_x(\phi_\epsilon(v) \partial_x v) = 0. \end{cases} \quad (6.11)$$

In order to obtain effective energy estimates, we take inspiration from Dalibard et al. 2019; Vasseur et al. 2016 and choose to re-write this system in terms of the integrated variables. Suppose we have initial data (w_0, v_0) such that $(w_0 - (w_\epsilon)(0), v_0 - (v_\epsilon)(0)) \in (L_0^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))^2$ where $L_0^1(\mathbb{R})$ is the subset of $L^1(\mathbb{R})$ consisting of zero mean functions. Then we define the integrated initial data as

$$W_0(x) := \int_{-\infty}^x (w_0(z) - w_\epsilon(0, z)) dz, \quad V_0(x) := \int_{-\infty}^x (v_0(z) - v_\epsilon(0, z)) dz.$$

Supposing that $(w - w_\epsilon, v - v_\epsilon)(t) \in L_0^1(\mathbb{R})$ holds true for positive times, we may also define the integrated variables

$$W(t, x) := \int_{-\infty}^x (w(t, z) - w_\epsilon(t, z)) dz, \quad V(t, x) := \int_{-\infty}^x (v(t, z) - v_\epsilon(t, z)) dz. \quad (6.12)$$

6. Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet
de suspension – 1. Introduction

Integrating (6.11) between $-\infty$ and x formally, we see that (W, V) solves

$$\begin{cases} \partial_t W = 0, \\ \partial_t V - \partial_x W - \phi_\epsilon(v) \partial_x v + \phi_\epsilon(v_\epsilon) \partial_x v_\epsilon = 0, \\ (W, V)(0, \cdot) = (W_0, V_0). \end{cases} \quad (6.13)$$

Since W is constant in time, we will denote it by its initial value W_0 from now on. The following result pertains to this system.

Theorem 6.1 (Existence of a strong solution to the integrated system). *Let $T > 0$ and $\gamma \geq 1$. Assume that $(W_0, V_0) \in (H^2(\mathbb{R}))^2$ and define*

$$\eta_0 := \frac{\partial_x V_0}{v_\epsilon|_{t=0} - 1}. \quad (6.14)$$

Suppose that $\eta_0 \in H^1(\mathbb{R})$, $\sqrt{x} \partial_x^k W_0 \in L^2(\mathbb{R}_+)$ for $k = 0, 1, 2$, and that the following estimate holds:

$$\begin{aligned} & \sum_{k=0}^2 \left(c_k \epsilon^{2k} E_k(0; V_2) + \epsilon^{2k-1} \|\sqrt{x} \partial_x^k W_0\|_{L^2(\mathbb{R}_+)}^2 \right) + \|W_0\|_{L^2(\mathbb{R})}^2 \\ & + \left(\frac{T}{\epsilon} \right)^{1/\gamma} \left(\epsilon^2 \|\partial_x W_0\|_{L^2(\mathbb{R})}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L^2(\mathbb{R})}^2 \right) \leq \delta_0 \epsilon^3 \end{aligned} \quad (6.15)$$

for some constants δ_0, c_0, c_1, c_2 depending only on s, γ, v_+ . Then there exists a unique solution (W_0, V) to (6.13) on $(0, T)$ such that

$$V \in C([0, T]; H^2(\mathbb{R})) \cap L^2(0, T; H^3(\mathbb{R})).$$

Moreover, there exists a constant $C = C(s, v_+, \gamma, \delta_0)$ such that

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\int_{\mathbb{R}} V^2 dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2 dx ds \right) \\ & + \sup_{t \in [0, T]} \left(\sum_{k=0}^1 \epsilon^{2k+2} \left[\int_{\mathbb{R}} \left| \partial_x^k \left(\frac{\partial_x V}{v_\epsilon - 1} \right) \right|^2 dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \left| \partial_x^{k+1} \left(\frac{\partial_x V}{v_\epsilon - 1} \right) \right|^2 dx ds \right] \right) \leq C \epsilon^3. \end{aligned} \quad (6.16)$$

Remark 6.1. *It follows from Proposition 6.1 that the coefficient of diffusion $\phi_\epsilon(v_\epsilon)$ (see (6.6) above) is singular and tends to $+\infty$ as x tends to $-\infty$. As a consequence, we cannot close the estimates on V in the classical Sobolev spaces H^s . This is why we work with the weighted quantity $\eta := \partial_x V / (v_\epsilon - 1)$ instead (see (6.35)), which yields better estimates. The drawback of this method is that the time-independent quantity W_0 cannot be bounded uniformly in time in such weighted spaces. Hence we only obtain local in time well-posedness when $W_0 \neq 0$, with a time of existence T proportional to $1 / \|\partial_x W_0\|_{H^1}^{2\gamma}$.*

6. Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet
de suspension – 1. Introduction

Remark 6.2. The assumption $\sqrt{x}\partial_x^k W_0 \in L^2(\mathbb{R}_+)$ for $k = 0, 1, 2$ is classical for this kind of system and was already used in Dalibard et al. 2021. However, it is possible to remove this assumption and only assume that $W_0 \in H^2(\mathbb{R})$. One then obtains a shorter time of existence, proportional to $1/\|W_0\|_{H^2}^2$ (see Remark 6.13 below).

Remark 6.3. Using $v_\epsilon < v_+$ and the regularity of v_ϵ , the bound (6.16) implies that $V \in L^\infty(0, T; H^2(\mathbb{R})) \cap L^2(0, T; H^3(\mathbb{R}))$. Furthermore, by using the smallness assumption (6.15), we will show the lower bound $v > 1$ for every t, x . In other words, the perturbation does not reach the congested state (see Remark 6.5).

Under the same assumptions, we also prove a stability result.

Theorem 6.2 (Nonlinear stability of travelling wave solutions). *Let $T > 0, \gamma \geq 1$. Assume that the initial data (u_0, v_0) is such that*

$$u_0 - (u_\epsilon)_{t=0} \in W_0^{1,1}(\mathbb{R}) \cap H^1(\mathbb{R}), \quad \frac{\partial_x[u_0 - (u_\epsilon)_{t=0}]}{v_\epsilon - 1} \in L^2(\mathbb{R}), \quad v_0 - (v_\epsilon)_{t=0} \in W_0^{2,1}(\mathbb{R}) \cap H^2(\mathbb{R}), \quad (6.17)$$

and the associated integrated initial data (W_0, V_0) satisfies (6.15). Then there exists a unique global solution (u, v) to (6.1) on $[0, T]$ which satisfies

$$\begin{aligned} u - u_\epsilon &\in C([0, T]; H^1(\mathbb{R}) \cap L_0^1(\mathbb{R})), \\ v - v_\epsilon &\in C([0, T]; H^1(\mathbb{R}) \cap L_0^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R})). \end{aligned}$$

In particular, there exists a constant $C_1 > 0$ dependent on γ, v_+, T such that

$$\|u - u_\epsilon\|_{L^\infty(0, T; H^1(\mathbb{R}))} + \|v - v_\epsilon\|_{L^\infty(0, T; H^1(\mathbb{R}))} + \|v - v_\epsilon\|_{L^2(0, T; H^2(\mathbb{R}))} \leq C_1.$$

There also exists a constant $C_2 > 0$ dependent on γ, v_+, T, ϵ such that

$$\|u - u_\epsilon\|_{L^\infty(0, T; L^1(\mathbb{R}))} + \|v - v_\epsilon\|_{L^\infty(0, T; L^1(\mathbb{R}))} \leq C_2.$$

As a consequence of the estimates derived during the course of proving the above theorem, we are also able to assert global existence with $T = +\infty$ and long-time stability if we remove the perturbation on the desired velocity.

Corollary 6.1. *If we additionally assume that*

$$(u - u_\epsilon)(0) = (\partial_x \phi_\epsilon(v) - \partial_x \phi_\epsilon(v_\epsilon))(0), \quad (6.18)$$

i.e. that $(w - w_\epsilon)(0) = 0$, then the solution (u, v) is defined on $\mathbb{R}_+ \times \mathbb{R}$, satisfies (6.16) with $T = +\infty$ and additionally

$$\sup_{x \in \mathbb{R}} (|v - v_\epsilon|(t, x) + |u - u_\epsilon|(t, x)) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

6. Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 2. Properties of the travelling wave solutions

Remark 6.4. All of our results remain true if we replace (6.2) by

$$w = u + \partial_y \tilde{p}_\epsilon(\rho) = u + \partial_y \left(\frac{\epsilon f(\rho)}{(1-\rho)^\gamma} \right), \quad \gamma \geq 1,$$

where f is smooth on $\mathbb{R}_+^*$ and such that $\gamma f(\rho) + (1-\rho)f'(\rho) > 0$, i.e. such that $\tilde{p}'_\epsilon(\rho) > 0$. In this case, one obtains that

$$\phi_\epsilon(v) = \epsilon v^{\gamma-3} \frac{(v-1)f'(1/v) + \gamma f(1/v)}{(v-1)^{\gamma+1}}. \quad (6.19)$$

In order to improve readability and without loss of generality, we will stick to the case $f(\rho) = \rho^\gamma$, for which the coefficient ϕ_ϵ can be written in the more compact form (6.6). This is also the form considered in Chaudhuri, Navoret, et al. 2024. All computations are similar (but heavier) in the general case (6.19).

2. Properties of the travelling wave solutions

2.1. Existence and asymptotic behavior of the travelling waves

We give here the proof of Proposition 6.1.

Proof. We first prove the existence of travelling wave solutions to (6.5):

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u - \partial_x (\phi_\epsilon(v) \partial_x u) = 0, \end{cases}$$

We look for a travelling wave solution of this system, i.e. a pair (u_ϵ, v_ϵ) where u_ϵ, v_ϵ are functions of the variable $\xi = x - st$, s being the speed of propagation of the solution. We also suppose that $(u_\epsilon, v_\epsilon) \rightarrow (u_\pm, v_\pm)$ and that $(u'_\epsilon, v'_\epsilon) \rightarrow (0, 0)$ as ξ goes to $\pm\infty$. We first obtain

$$\begin{cases} -sv'_\epsilon - u'_\epsilon = 0, \\ -su'_\epsilon - (\phi_\epsilon(v_\epsilon)u'_\epsilon)' = 0. \end{cases}$$

Integrating this equation between ξ and $+\infty$, we get

$$\begin{cases} sv_\epsilon + u_\epsilon = sv_+ + u_+, \\ su_\epsilon + \phi_\epsilon(v_\epsilon)u'_\epsilon = su_+. \end{cases}$$

The first equation yields $u_\epsilon = sv_+ + u_+ - sv_\epsilon$. Substituting this into the second line, we deduce that v_ϵ solves the following ODE:

$$v'_\epsilon = \frac{s(v_+ - v_\epsilon)}{\phi_\epsilon(v_\epsilon)} = \frac{s(v_+ - v_\epsilon)v_\epsilon(v_\epsilon - 1)^{\gamma+1}}{\epsilon\gamma}. \quad (6.20)$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 2. Properties of the travelling wave solutions*

Assuming $v_+ > 1$ and $s > 0$, we may deduce that $v_- = 1$, and $u_- = s(v_+ - 1) + u_+$, or equivalently $s = (u_- - u_+)/ (v_+ - 1)$. The function v_ϵ is therefore an increasing function taking values in the interval $(1, v_+)$ (see Figure 6.1). The Cauchy-Lipschitz theorem thus yields that v_ϵ is the unique (up to a translation) global solution of (6.20), as stated in Proposition 6.1. Note that, as v_ϵ approaches 1, the diffusion coefficient $\phi_\epsilon(v_\epsilon)$ tends to $+\infty$.

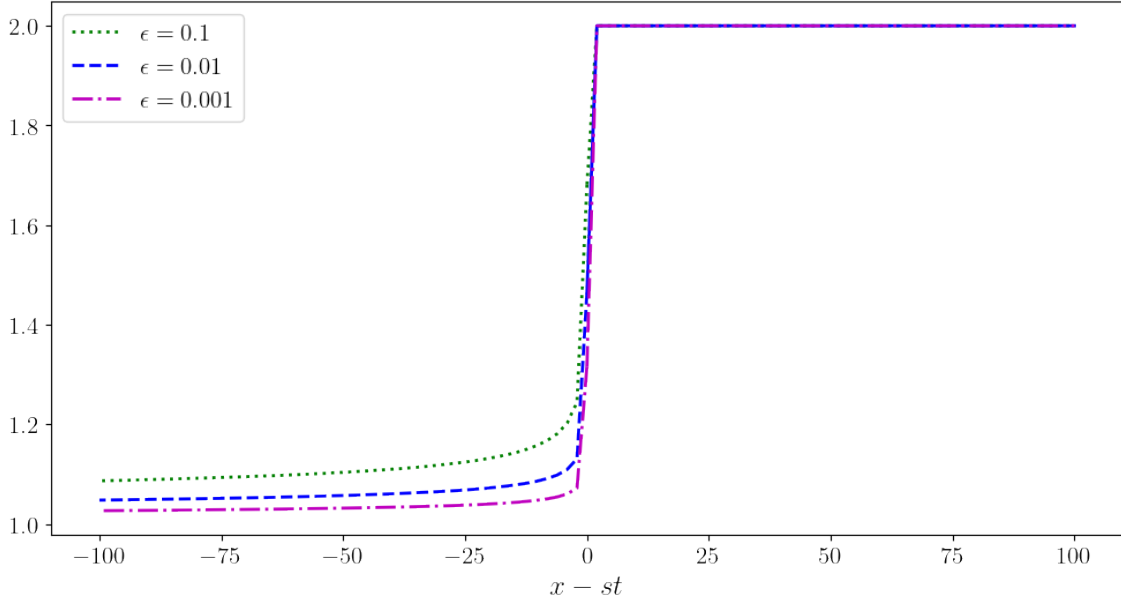


Figure 6.1. – The profile of v_ϵ , where we fix $v_\epsilon(0) = (1 + v_+)/2$ and $v_+ = 2, \gamma = 5$.

Let us now make this statement more quantitative. From now on, suppose that $v_\epsilon(0) = (1 + v_+)/2$, i.e. v_ϵ is halfway between 1 and v_+ . Let $\xi < 0$. From the ODE (6.20) and the monotonicity of v_ϵ , we obtain the following bound:

$$\frac{s(v_+ - 1)(v_\epsilon(\xi) - 1)^{\gamma+1}}{2\epsilon\gamma} \leq v'_\epsilon(\xi) \leq \frac{s(v_+ - 1)(v_+ + 1)(v_\epsilon(\xi) - 1)^{\gamma+1}}{2\epsilon\gamma}.$$

Dividing by $(v_\epsilon(\xi) - 1)^{\gamma+1}$ and integrating between ξ and 0 yields

$$-\frac{s(v_+ - 1)}{2\epsilon\gamma}\xi \leq -\frac{2^\gamma}{\gamma(v_+ - 1)^\gamma} + \frac{1}{\gamma(v_\epsilon(\xi) - 1)^\gamma} \leq -\frac{s(v_+ - 1)(v_+ + 1)}{2\epsilon\gamma}\xi,$$

i.e.

$$1 + \left(B - \frac{A_0}{\epsilon}\xi\right)^{-1/\gamma} \leq v_\epsilon(\xi) \leq 1 + \left(B - \frac{A_1}{\epsilon}\xi\right)^{-1/\gamma},$$

where

$$A_0 := \frac{s(v_+ - 1)(v_+ + 1)}{2}, \quad A_1 := \frac{s(v_+ - 1)}{2}, \quad \text{and } B := \left(\frac{2}{v_+ - 1}\right)^\gamma.$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 2. Properties of the travelling wave solutions*

Estimate (6.10) follows. When $\xi > 0$, one has similarly

$$\frac{s(\nu_+ + 1)(\nu_+ - 1)^{\gamma+1}}{2\gamma^2\epsilon\gamma} \leq \frac{\nu'_\epsilon(\xi)}{\nu_+ - \nu_\epsilon(\xi)} \leq \frac{s\nu_+(\nu_+ - 1)^{\gamma+1}}{\epsilon\gamma},$$

i.e.

$$\frac{A_2}{\epsilon} \leq \frac{\nu'_\epsilon(\xi)}{\nu_+ - \nu_\epsilon(\xi)} \leq \frac{A_3}{\epsilon},$$

with

$$A_2 := \frac{s(\nu_+ + 1)(\nu_+ - 1)^{\gamma+1}}{2\gamma^2\gamma} \quad \text{and} \quad A_3 := \frac{s\nu_+(\nu_+ - 1)^{\gamma+1}}{\gamma}.$$

Integrating between 0 and ξ yields

$$\frac{A_2}{\epsilon}\xi \leq -\ln[\nu_+ - \nu_\epsilon(\xi)] + \ln\left[\frac{\nu_+ - 1}{2}\right] \leq \frac{A_3}{\epsilon}\xi,$$

i.e.

$$\frac{\nu_+ - 1}{2} \exp\left(-\frac{A_2}{\epsilon}\xi\right) \geq \nu_+ - \nu_\epsilon(\xi) \geq \frac{\nu_+ - 1}{2} \exp\left(-\frac{A_3}{\epsilon}\xi\right),$$

which is the desired estimate. □

2.2. Passage to the integrated system and reformulation of the equations

We now want to study the stability of the travelling wave solution obtained in the previous section. In order to do so, we first rewrite (6.5) in term of the unknowns ν and $w = u - \phi_\epsilon(\nu)\partial_x \nu$:

$$\begin{cases} \partial_t w = 0, \\ \partial_t \nu - \partial_x w - \partial_x(\phi_\epsilon(\nu)\partial_x \nu) = 0. \end{cases} \quad (6.21)$$

Let $(w_\epsilon, \nu_\epsilon)$ denote the travelling wave solution. It solves the same system:

$$\begin{cases} \partial_t w_\epsilon = 0, \\ \partial_t \nu_\epsilon - \partial_x w_\epsilon - \partial_x(\phi_\epsilon(\nu_\epsilon)\partial_x \nu_\epsilon) = 0. \end{cases} \quad (6.22)$$

Taking the difference between these two systems yields

$$\begin{cases} \partial_t(w - w_\epsilon) = 0, \\ \partial_t(\nu - \nu_\epsilon) - \partial_x(w - w_\epsilon) - \partial_x(\phi_\epsilon(\nu)\partial_x \nu) + \partial_x(\phi_\epsilon(\nu_\epsilon)\partial_x \nu_\epsilon) = 0. \end{cases} \quad (6.23)$$

Following Dalibard et al. 2019; Vasseur et al. 2016, we introduce W, V such that

$$W(t, x) := \int_{-\infty}^x (w(t, x') - w_\epsilon(t, x')) dx' \quad \text{and} \quad V(t, x) := \int_{-\infty}^x (\nu(t, x') - \nu_\epsilon(t, x')) dx'. \quad (6.24)$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 2. Properties of the travelling wave solutions*

Integrating (6.23) between $-\infty$ and x yields for (W, V) the following system:

$$\begin{cases} \partial_t W = 0, \\ \partial_t V - \partial_x W - \phi_\epsilon(v) \partial_x v + \phi_\epsilon(v_\epsilon) \partial_x v_\epsilon = 0. \end{cases}$$

Introducing ψ_ϵ such that $\psi'_\epsilon = \phi_\epsilon$, we obtain that $W = W_0$ is constant and

$$\partial_t V - \partial_x W_0 - \partial_x(\psi_\epsilon(v) - \psi_\epsilon(v_\epsilon)) = 0. \quad (6.25)$$

We first notice that we can replace v by $v_\epsilon + \partial_x V$. In order to write the system as a linearized part around v_ϵ plus a perturbation part, we also move the terms involving ψ_ϵ to the right-hand side and subtract $\partial_x(\phi_\epsilon(v_\epsilon) \partial_x V)$ from both sides:

$$\partial_t V - \partial_x W_0 - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x V) = \partial_x [\psi_\epsilon(v_\epsilon + \partial_x V) - \psi_\epsilon(v_\epsilon) - \psi'_\epsilon(v_\epsilon) \partial_x V].$$

This previous equation can be written in the following compact way:

$$\partial_t V - \partial_x W_0 - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x V) = \partial_x H(\partial_x V), \quad (6.26)$$

with

$$H(f) := \psi_\epsilon(v_\epsilon + f) - \psi_\epsilon(v_\epsilon) - \psi'_\epsilon(v_\epsilon) f. \quad (6.27)$$

2.3. Preliminary estimates

We give in this section some estimates on the functions v_ϵ , ψ_ϵ , H and their derivatives.

Lemma 6.1. *(Estimates on v_ϵ) For any $k \geq 1$, there exists a constant $C = C(k, \gamma, v_+, s)$ such that*

$$\forall \epsilon > 0, \quad \left| \partial_x^k v_\epsilon \right| \leq \frac{C(v_\epsilon - 1)^{k\gamma+1}}{\epsilon^k}. \quad (6.28)$$

Proof. By using the ODE (6.20) satisfied by v_ϵ , one can show by induction that for every $k \geq 1$, there exists a function f_k independent of ϵ such that

- f_k is smooth on $\mathbb{R}_+$,
- $f_k(1) \neq 0$,
- $\partial_x^k v_\epsilon = f_k(v_\epsilon)(v_\epsilon - 1)^{k\gamma+1}/\epsilon^k$.

Since f_k is continuous on $[1, v_+]$ and $1 < v_\epsilon < v_+$, the factor $f_k(v_\epsilon)$ is uniformly bounded. $\square$

Lemma 6.2. *(Estimates on ψ_ϵ) Let $\bar{v} > 1$ be arbitrary. For any $k \geq 1$, there exists $C = C(k, \gamma, \bar{v})$ such that, for every $v \in (1, \bar{v})$, for every $\epsilon > 0$,*

$$\left| \psi_\epsilon^{(k)}(v) \right| \leq \frac{C\epsilon}{(v-1)^{\gamma+k}}. \quad (6.29)$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 2. Properties of the travelling wave solutions*

Proof. As in the previous lemma, one can show that there exists a smooth function g_k defined on $\mathbb{R}_+^*$, independent of ϵ , such that $\psi_\epsilon^{(k)}(v) = \epsilon g_k(v)(v-1)^{-\gamma-k}$ and $g_k(1) \neq 0$. Since g_k is smooth, we can bound it on $[1, \bar{v}]$. $\square$

As a consequence of Lemmas 6.1 and 6.2, we obtain the following estimates:

Lemma 6.3. *The functions $\phi_\epsilon^{1/2} \partial_x v_\epsilon$ and $\phi_\epsilon^{1/2} \frac{\partial_x^k v_\epsilon}{v_\epsilon - 1}$, $k \geq 2$, belong to $L^2(\mathbb{R})$, with a time-independent $L^2(\mathbb{R})$ norm.*

Proof. We first see with Proposition 6.1 and the ODE (6.20) satisfied by v_ϵ that, for any $k \geq 1$, $\partial_x^k v_\epsilon \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Since ϕ_ϵ and $v_\epsilon - 1$ go to a positive and finite limit as ξ goes to $+\infty$, the integrability of the functions $\phi_\epsilon^{1/2} \partial_x v_\epsilon$ and $\phi_\epsilon^{1/2} \frac{\partial_x^k v_\epsilon}{v_\epsilon - 1}$ at $+\infty$ is a consequence of the one of $\partial_x^k v_\epsilon$.

Concerning the integrability when $\xi \rightarrow -\infty$, we first see with Lemma 6.1 that there exists $C = C(\epsilon) > 0$ such that

$$|\phi_\epsilon^{1/2} \partial_x v_\epsilon| \leq C(v_\epsilon - 1)^{(\gamma+1)/2}, \quad \text{and} \quad \left| \phi_\epsilon^{1/2} \frac{\partial_x^k v_\epsilon}{v_\epsilon - 1} \right| \leq C(v_\epsilon - 1)^{(2k-1)\gamma/2-1/2}.$$

By Proposition 6.1, we deduce that

$$|\phi_\epsilon^{1/2} \partial_x v_\epsilon| \leq C \left(B - \frac{A_1}{\epsilon} \xi \right)^{-1/2-1/(2\gamma)}$$

and

$$\left| \phi_\epsilon^{1/2} \frac{\partial_x^k v_\epsilon}{v_\epsilon - 1} \right| \leq C \left(B - \frac{A_1}{\epsilon} \xi \right)^{1/2+1/(2\gamma)-k} \in L^2(\mathbb{R}_-)$$

since $k \geq 2$. Note that we consider functions that depend only on the variable $\xi = x - st$, and so the L^2 -norms are independent of time. $\square$

Lemma 6.4. *(Bounds on H) Let f such that there exists $\delta < 1$ with $\| \frac{f}{v_\epsilon - 1} \|_\infty \leq \delta$. Then one has the following bounds on $H(f)$:*

$$|H(f)| \leq \frac{C\epsilon}{(v_\epsilon - 1)^{\gamma+2}} f^2, \tag{6.30}$$

$$|\partial_x H(f)| \leq \frac{C}{(v_\epsilon - 1)^2} f^2 + \frac{C\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |f| |\partial_x f|, \tag{6.31}$$

$$|\partial_x^2 H(f)| \leq \frac{C(v_\epsilon - 1)^{\gamma-2}}{\epsilon} f^2 + \frac{C\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |f| |\partial_x^2 f| + \frac{C\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x f|^2, \tag{6.32}$$

for some $C = C(s, \gamma, \delta, v_+)$.

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

Proof. We first prove (6.30). Recall that

$$H(f) = \psi_\epsilon(v_\epsilon + f) - \psi_\epsilon(v_\epsilon) - \psi'_\epsilon(v_\epsilon)f.$$

By Taylor's theorem,

$$|H(f)| \leq \frac{f^2}{2} \sup_{|v-v_\epsilon| \leq |f|} |\psi_\epsilon^{(2)}(v)|.$$

By the hypothesis on f , for any v such that $|v - v_\epsilon| \leq |f|$, it holds $0 < (1 - \delta)(v_\epsilon - 1) < v - 1 < 2v^+$. Lemma 6.2 then implies that $|\psi_\epsilon^{(2)}(v)| \leq C\epsilon(v - 1)^{-(\gamma+2)}$. Hence

$$|H(f)| \leq \frac{C\epsilon}{(v_\epsilon - 1)^{\gamma+2}} f^2,$$

for some constant C depending on γ, v_+, δ . This is the first inequality. The other inequalities are proved in the same way. We differentiate H with respect to x and obtain

$$\partial_x H(f) = \partial_x v_\epsilon (\psi'_\epsilon(v_\epsilon + f) - \psi'_\epsilon(v_\epsilon) - \psi_\epsilon^{(2)}(v_\epsilon)f) + \partial_x f (\psi'_\epsilon(v_\epsilon + f) - \psi'_\epsilon(v_\epsilon)).$$

which yields (6.31) by the same arguments and Lemmas 6.1, 6.2. We now differentiate a second time with respect to x :

$$\begin{aligned} \partial_{xx} H(f) = & \partial_{xx} v_\epsilon (\psi'_\epsilon(v_\epsilon + f) - \psi'_\epsilon(v_\epsilon) - \psi_\epsilon^{(2)}(v_\epsilon)f) + (\partial_x f)^2 \psi_\epsilon^{(2)}(v_\epsilon + f) \\ & + (\partial_x v_\epsilon)^2 (\psi_\epsilon^{(2)}(v_\epsilon + f) - \psi_\epsilon^{(2)}(v_\epsilon) - \psi_\epsilon^{(3)}(v_\epsilon)f) \\ & + 2\partial_x f \partial_x v_\epsilon (\psi_\epsilon^{(2)}(v_\epsilon + f) - \psi_\epsilon^{(2)}(v_\epsilon)) + \partial_{xx} f (\psi'_\epsilon(v_\epsilon + f) - \psi'_\epsilon(v_\epsilon)). \end{aligned}$$

which yields (6.32) by similar computations. $\square$

3. Well-posedness for the integrated system

3.1. Basic energy estimate

The goal of this section is to prove the existence of strong solutions to the equation

$$\partial_t V - \partial_x W_0 - \partial_x (\phi_\epsilon(v_\epsilon) \partial_x V) = \partial_x H(\partial_x V). \quad (6.33)$$

Since W_0 is constant in time, we can control it for positive times through suitable assumptions on the initial data. Thus without loss of generality we treat W_0 as a perturbation of the system and prove the existence of a strong solution V to (6.33). Our strategy is to employ a fixed-point argument, which requires us to first derive appropriate energy estimates. The first (zero-th order) estimate is obtained by multiplying

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

(6.33) by V :

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} V^2(t) dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2 dx d\tau \\ &= \frac{1}{2} \int_{\mathbb{R}} V^2(0) dx + \int_0^t \int_{\mathbb{R}} V(\partial_x H(\partial_x V) + \partial_x W_0) dx d\tau. \end{aligned} \quad (6.34)$$

From the left-hand side of the equation, we see that the natural energy space is $V \in L^\infty([0, T], L^2(\mathbb{R}))$ and $\sqrt{\phi_\epsilon(v_\epsilon)} \partial_x V \in L^2((0, T) \times \mathbb{R})$. Let us now move to the right-hand side of (6.34) and try to close the estimate. For the term containing W_0 , an integration by parts and Young's inequality yield

$$\left| \int_0^t \int_{\mathbb{R}} V \partial_x W_0 \right| = \left| - \int_0^t \int_{\mathbb{R}} W_0 \partial_x V \right| \leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2 + \frac{1}{2} \int_0^t \int_{\mathbb{R}} \frac{W_0^2}{\phi_\epsilon(v_\epsilon)}.$$

Hence this term can be bounded through suitable assumptions on W_0 . For the term containing $H(\partial_x V)$, we also perform an integration by parts and use the estimate (6.30) on H that we computed before:

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} V \partial_x H(\partial_x V) \right| &= \left| - \int_0^t \int_{\mathbb{R}} H(\partial_x V) \partial_x V \right| \\ &\leq C\epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x V|^3}{(v_\epsilon - 1)^{\gamma+2}} \\ &\leq C\epsilon \left\| \frac{1}{\phi_\epsilon(v_\epsilon)(v_\epsilon - 1)^{\gamma+1}} \right\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2 \frac{|\partial_x V|}{v_\epsilon - 1}. \end{aligned}$$

By recalling that $\phi_\epsilon(v_\epsilon) = \frac{\epsilon\gamma}{v_\epsilon(v_\epsilon - 1)^{\gamma+1}}$ (see (6.6)), we obtain that

$$\left| \int_0^t \int_{\mathbb{R}} V \partial_x H(\partial_x V) \right| \leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2 \frac{|\partial_x V|}{v_\epsilon - 1} \leq C \left\| \frac{\partial_x V}{v_\epsilon - 1} \right\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V|^2,$$

where $C = C(s, \gamma, \delta)$. In order to close the estimate, we need that the norm of the quantity $\partial_x V / (v_\epsilon - 1)$ in $L^\infty([0, T] \times \mathbb{R})$ be small enough. The ideas of the proof of the well-posedness of (6.33) are thus the following:

- Derive a-priori energy estimates for the quantities $\partial_x V / (v_\epsilon - 1)$ and $\partial_x [\partial_x V / (v_\epsilon - 1)]$.
- Use the fact that $\partial_x V / (v_\epsilon - 1) \in L^\infty([0, T], H^1(\mathbb{R}))$ and the injection $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ to complete the estimates and bound these quantities in the same function spaces as chosen for V .
- Use a fixed-point argument to obtain the local-in-time existence and uniqueness of strong solutions to (6.33).

Remark 6.5. *Note that the assumption that $\|\partial_x V / (v_\epsilon - 1)\|_{L_{t,x}^\infty}$ is small, which enables to close the previous estimate, is also the one needed to bound the quantity $H(\partial_x V)$ and its derivatives (see Lemma 6.4). This assumption has a simple interpretation; by recalling that $\partial_x V = v - v_\epsilon$, we see that it prevents the perturbation v from reaching 1.*

6. Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system

In other words, this assumption ensures that $\rho < 1$, i.e. that the perturbation stays in the uncongested state.

Remark 6.6. As explained in the previous remark, it is crucial to bound the quantity $\partial_x V / (v_\epsilon - 1)$. Note that adding the factor $1 / (v_\epsilon - 1)$ is not free, and the price to pay is that a lot of commutators appear and the equations become more complicated (see the next subsection). However, even if the weight $1 / (v_\epsilon - 1)$ is singular, the commutators are not difficult to bound. For instance,

$$\partial_x \left(\frac{\partial_x V}{v_\epsilon - 1} \right) = \frac{\partial_x^2 V}{v_\epsilon - 1} - \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \partial_x V.$$

From the estimate on v'_ϵ that we derived in Lemma (6.1), it follows that the coefficient $v'_\epsilon / (v_\epsilon - 1)^2$ is bounded by $C(v_\epsilon - 1)^{\gamma-1} / \epsilon$. Hence this coefficient is uniformly bounded in t, x as soon as $\gamma \geq 1$, which is our hypothesis throughout the paper. The computations of the next subsections show that $\gamma \geq 1$ is the only hypothesis needed in order to obtain good bounds for every quantity which appears due to commutators. Furthermore, the factor $1/\epsilon$ in the coefficient $v'_\epsilon / (v_\epsilon - 1)^2$ of the commutator suggests that the operator ∂_x scales as $1/\epsilon$ for solutions of the equations, i.e. that $\|\partial_x f\| \cong \|f\| / \epsilon$. This observation is reflected in the definition of the norm to be seen in Section 3.3.

3.2. Formulation of the system for higher order estimates

In this subsection we describe our approach for deriving higher order energy estimates. Our estimates will heavily involve the quantity

$$\eta \equiv \eta(\partial_x V) := \frac{\partial_x V}{v_\epsilon - 1}. \quad (6.35)$$

The evolution equation for η reads as

$$\begin{aligned} \partial_t \eta - \partial_x (\phi_\epsilon(v_\epsilon) \partial_x \eta) &= \frac{\partial_x^2 W_0}{v_\epsilon - 1} + \frac{\partial_x^2 H}{v_\epsilon - 1} + \frac{s \eta v'_\epsilon}{v_\epsilon - 1} \\ &\quad + \frac{\phi_\epsilon(v_\epsilon) v'_\epsilon \partial_x^2 V}{(v_\epsilon - 1)^2} + \partial_x \left(\frac{\eta \phi_\epsilon(v_\epsilon) v'_\epsilon}{v_\epsilon - 1} \right) + \frac{1}{v_\epsilon - 1} \partial_x (\phi'_\epsilon(v_\epsilon) v'_\epsilon \partial_x V), \end{aligned} \quad (6.36)$$

which may be expressed as

$$\partial_t \eta - \partial_x (\phi_\epsilon(v_\epsilon) \partial_x \eta) = \mathcal{L}_\epsilon(\eta) + S_\epsilon(\partial_x V), \quad (6.37)$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

where

$$\begin{aligned}\mathcal{L}_\epsilon(\eta) &:= \frac{s\eta v'_\epsilon}{v_\epsilon - 1} + \frac{\phi_\epsilon(v_\epsilon)v'_\epsilon}{v_\epsilon - 1} \left(\partial_x \eta + \frac{\eta v'_\epsilon}{v_\epsilon - 1} \right) + \partial_x \left(\frac{\eta \phi_\epsilon(v_\epsilon)v'_\epsilon}{v_\epsilon - 1} \right) + \frac{1}{v_\epsilon - 1} \partial_x (\phi'_\epsilon(v_\epsilon)v'_\epsilon(v_\epsilon - 1)\eta), \\ S_\epsilon(\partial_x V) &:= \frac{\partial_x^2 [W_0 + H(\partial_x V)]}{v_\epsilon - 1}.\end{aligned}$$

Differentiating (6.37), we obtain

$$\partial_t(\partial_x \eta) - \partial_x(\phi_\epsilon(v_\epsilon)\partial_x^2 \eta) = \partial_x S_\epsilon(\partial_x V) + \mathcal{L}_\epsilon(\partial_x \eta) + \mathcal{C}_\epsilon(\eta), \quad (6.38)$$

where $\mathcal{C}_\epsilon(\eta) := [\partial_x, \partial_x(\phi_\epsilon \partial_x \cdot)]\eta + [\partial_x, \mathcal{L}_\epsilon]\eta = \partial_x(\partial_x \phi_\epsilon(v_\epsilon)\partial_x \eta) + [\partial_x, \mathcal{L}_\epsilon]\eta$ appears due to commutators and

$$\begin{aligned}[\partial_x, \mathcal{L}_\epsilon]\eta &:= s\eta \partial_x \left(\frac{v'_\epsilon}{v_\epsilon - 1} \right) + \partial_x \left(\frac{\phi_\epsilon(v_\epsilon)v'_\epsilon}{v_\epsilon - 1} \right) \partial_x \eta \\ &\quad + \partial_x \left(\phi_\epsilon(v_\epsilon)(v'_\epsilon)^2 \frac{1}{(v_\epsilon - 1)^2} \right) \eta + \partial_x \left(\eta \partial_x \left(\frac{\phi_\epsilon(v_\epsilon)v'_\epsilon}{v_\epsilon - 1} \right) \right) \\ &\quad - \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \partial_x (\phi'_\epsilon v'_\epsilon (v_\epsilon - 1)\eta) + \frac{1}{v_\epsilon - 1} \partial_x (\partial_x (\phi'_\epsilon(v_\epsilon)v'_\epsilon(v_\epsilon - 1))\eta).\end{aligned}$$

The equations (6.37) and (6.38) will be used to derive key estimates that allow us to eventually conclude the fixed-point argument.

Note that the operators $\mathcal{L}_\epsilon$ and $\mathcal{C}_\epsilon$, that appear in equations (6.37) and (6.38), are linear. In order to improve readability, we give bounds for these two operators in the following lemma.

Lemma 6.5. *There exists $C = C(\gamma, s, v_+) > 0$ such that, for every $\alpha \in (0, 1)$, for every $\epsilon > 0$, for any η sufficiently regular, there holds*

$$\left| \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\eta) \eta \right| \leq \alpha \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) (\partial_x \eta)^2 + \frac{C}{\alpha \epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) (\partial_x V)^2 \quad (6.39)$$

and

$$\left| \int_0^t \int_{\mathbb{R}} \mathcal{C}_\epsilon(\eta) \partial_x \eta \right| \leq \alpha \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) (\partial_x^2 \eta)^2 + \frac{C}{\alpha \epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \left((\partial_x \eta)^2 + \frac{(\partial_x V)^2}{\epsilon^2} \right). \quad (6.40)$$

We defer the proof of this lemma to Appendix 5.2.

3.3. Construction of global strong solutions

In this subsection we lay out the framework for our fixed point argument before finally stating the result of this section, i.e. the well posedness of (6.33) in an appropriate

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

space. Firstly, we fix $T > 0$ arbitrary. For $t \in [0, T]$, we define the energies

$$E_0(t; V) := \int_{\mathbb{R}} V^2(t) dx, \quad D_0(t; V) := \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V(t)|^2 dx, \quad (6.41)$$

as well as

$$E_k(t; V) := \int_{\mathbb{R}} |\partial_x^{k-1} \eta|^2(t) dx, \quad D_k(t; V) := \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x^k \eta|^2(t) dx, \quad \text{for } k = 1, 2. \quad (6.42)$$

Then given V_1 , we introduce the system

$$\begin{cases} \partial_t V_2 - \partial_x W_0 - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x V_2) = \partial_x H(\partial_x V_1), \\ V_2|_{t=0} = V_0, \end{cases} \quad (6.43)$$

and the application $\mathcal{A}^\epsilon : V_1 \mapsto V_2$, where $V_1, V_2 \in \mathcal{X}$ and

$$\mathcal{X} = \{V : E_k(t, V) \in L^\infty(0, T) \text{ and } D_k(t; V) \in L^1(0, T) \text{ for } k = 0, 1, 2.\},$$

$$\|V\|_{\mathcal{X}}^2 := \sup_{t \in [0, T]} \left(\sum_{k=0}^2 c_k \epsilon^{2k} \left[E_k(t; V(t)) + \int_0^t D_k(\tau; V(\tau)) d\tau \right] \right),$$

where $c_k = c_k(s, \gamma, v_+)$, for $k = 0, 1, 2$, are constants independent of ϵ which are to be determined. For a proof that the map $\mathcal{A}^\epsilon$ is well defined, see Appendix 5.1. For $\delta > 0$ we define the ball

$$B_\delta := \{V \in \mathcal{X} : \|V\|_{\mathcal{X}} < \delta \epsilon^{3/2}\}. \quad (6.44)$$

The remainder of this section aims to prove the following result.

Proposition 6.2. *Assume that for some $\delta_0 = \delta_0(s, v_+, \gamma) > 0$ it holds*

$$\begin{aligned} & \sum_{k=0}^2 \left(c_k \epsilon^{2k} E_k(0; V_2) + \epsilon^{2k-1} \|\sqrt{x} \partial_x^k W_0\|_{L^2(\mathbb{R}_+)}^2 \right) \\ & + \|W_0\|_{L_x^2}^2 + \left(\frac{T}{\epsilon} \right)^{1/\gamma} \left(\epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) \leq \delta_0 \epsilon^3, \end{aligned} \quad (6.45)$$

where $c_k = c_k(s, \gamma, v_+)$ for $k = 0, 1, 2$ are positive constants. Then there exists $\delta = \delta(s, \gamma, v_+)$ such that

1. the ball B_δ is stable by $\mathcal{A}^\epsilon$,
2. the map $\mathcal{A}^\epsilon$ is a contraction on B_δ .

Consequently, $\mathcal{A}^\epsilon$ has a unique fixed point in B_δ .

3.4. Stability of the ball B_δ

We let $\delta > 0$, $V_1 \in B_\delta$ and $V_2 := \mathcal{A}^\epsilon(V_1)$. The aim of this subsection is to show that we can find $\delta > 0$ such that $V_2 \in B_\delta$. First, let us clarify the meaning of some notation we will use in this section.

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

- We denote by $C = C(s, \gamma, \nu_+)$ an arbitrary positive constant independent of ϵ . This constant may change, even within the same line. We also denote $C' = C'(s, \gamma, \nu_+, c_0, c_1, c_2)$ another positive constant that may additionally depend on the constants c_0, c_1, c_2 appearing in the definition of the norm $\|\cdot\|_{\mathcal{X}}$.
- To ease notation, we will often shorten $\phi_\epsilon(\nu_\epsilon), \psi_\epsilon(\nu_\epsilon)$ to $\phi_\epsilon, \psi_\epsilon$ respectively. For $i = 1, 2$ we also adopt the notation $\eta_i := \frac{\partial_x V_i}{\nu_\epsilon - 1}$.
- We adopt the notation $X_t Y_x := X(0, T; Y(\mathbb{R}))$ for appropriate function spaces X and Y .

Next, we make note of some useful estimates which will be repeatedly used in the remainder of the paper. We have up to a constant independent of ϵ ,

$$\begin{aligned}\phi_\epsilon(\nu_\epsilon) &= \frac{\epsilon\gamma}{\nu_\epsilon(\nu_\epsilon - 1)^{\gamma+1}}, \\ \phi_\epsilon^{(k)}(\nu_\epsilon) &\cong \frac{\phi_\epsilon(\nu_\epsilon)}{(\nu_\epsilon - 1)^k},\end{aligned}\tag{6.46}$$

$$\partial_x \nu_\epsilon = \nu'_\epsilon \cong \frac{1}{\epsilon}(\nu_\epsilon - 1)^{\gamma+1}.\tag{6.47}$$

As a consequence of (6.46) and (6.47), we have $(\nu_\epsilon - 1)^2 \partial_x \nu_\epsilon \leq C/\epsilon$ and $(\nu_\epsilon - 1)^2 / \phi_\epsilon \leq C/\epsilon$, provided $\gamma \geq 1$. Let us now note a Gagliardo-Nirenberg-Sobolev interpolation inequality which we will use. For $f \in H^1(\mathbb{R})$,

$$\|f\|_{L_x^\infty} \leq C \|\partial_x f\|_{L_x^2}^{\frac{1}{2}} \|f\|_{L_x^2}^{\frac{1}{2}},\tag{6.48}$$

Finally, we mention some guidelines which our estimates will follow.

- We will often artificially place a factor of ϕ_ϵ into the integral in order to obtain an expression which is a function of the energies (6.41)-(6.42). This results in the multiplication of a factor of $1/\epsilon$. For example, to estimate the term $A := \int_0^t \int_{\mathbb{R}} \partial_x V \partial_x \eta \, dx d\tau$, we have

$$A \leq \left\| \frac{1}{\phi_\epsilon} \right\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} |\sqrt{\phi_\epsilon} \partial_x V| |\sqrt{\phi_\epsilon} \partial_x \eta| \, dx d\tau \leq \frac{C}{\epsilon} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L_{t,x}^2}.\tag{6.49}$$

- To estimate terms involving the expression $\partial_x^2 V / (\nu_\epsilon - 1)$, we will make use of the identity

$$\frac{\partial_x^2 V}{\nu_\epsilon - 1} = \partial_x \eta + \frac{\nu'_\epsilon \partial_x V}{(\nu_\epsilon - 1)^2}.\tag{6.50}$$

3.4.1. Estimates for $k = 0$

Fix $t \in [0, T]$. We multiply (6.43) by V_2 and integrate on $\mathbb{R} \times (0, t)$ to obtain as before:

$$\frac{1}{2} \int_{\mathbb{R}} V_2^2(t) \, dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x V_2|^2 \, dx d\tau = \frac{1}{2} \int_{\mathbb{R}} V_2^2(0) \, dx + \int_0^t \int_{\mathbb{R}} V_2 (\partial_x H(\partial_x V_1) + \partial_x W_0) \, dx d\tau.\tag{6.51}$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

Integrating by parts, we have

$$\int_0^t \int_{\mathbb{R}} V_2 \partial_x (H(\partial_x V_1) + W_0) = - \int_0^t \int_{\mathbb{R}} H(\partial_x V_1) \partial_x V_2 - \int_0^t \int_{\mathbb{R}} W_0 \partial_x V_2 =: G_1 + G_2. \quad (6.52)$$

Using the estimate on H and (6.48), and introducing $\eta_2 = \frac{\partial_x V_2}{v_\epsilon - 1}$,

$$\begin{aligned} |G_1| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \frac{|\partial_x V_1|^2}{v_\epsilon - 1} |\partial_x V_2| = C \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x V_1|^2 |\eta_2| \\ &\leq C \|\eta_2\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2}^2 \leq C \|\eta_2\|_{L_t^\infty L_x^2}^{1/2} \|\partial_x \eta_2\|_{L_t^\infty L_x^2}^{1/2} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2}^2 \\ &\leq \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \end{aligned} \quad (6.53)$$

Remark 6.7. *Note the following estimate, which can be obtained from (6.48) and will also be used later on:*

$$\|\eta\|_{L_{t,x}^\infty} \leq \frac{C'}{\epsilon^{3/2}} \|V\|_{\mathcal{X}}. \quad (6.54)$$

We now estimate G_2 . Young's inequality gives that

$$\left| \int_0^t \int_{\mathbb{R}} W_0 \partial_x V_2 \right| \leq \frac{1}{4} \int_0^t D_0(\tau; V_2) d\tau + \int_0^t \int_{\mathbb{R}} \frac{W_0^2}{\phi_\epsilon}.$$

Using Fubini and the fact that W_0 is independant of time, one then has

$$\int_0^t \int_{\mathbb{R}} \frac{W_0^2}{\phi_\epsilon} = \int_{\mathbb{R}} W_0^2 \int_0^t \frac{1}{\phi_\epsilon} = \int_{\mathbb{R}} W_0^2 \int_0^t \frac{1}{\phi_\epsilon} \mathbf{1}_{\xi < 0} d\tau dx + \int_{\mathbb{R}} W_0^2 \int_0^t \frac{1}{\phi_\epsilon} \mathbf{1}_{\xi > 0} d\tau dx,$$

where $\xi := x - s\tau$. By using Equation (6.10), one then obtain for the first integral the estimate

$$\int_{\mathbb{R}} W_0^2 \int_0^t \frac{1}{\phi_\epsilon} \mathbf{1}_{\xi < 0} d\tau dx \leq \frac{C}{\epsilon} \int_{\mathbb{R}} W_0^2 \int_0^{+\infty} \left(B - \frac{A_1}{\epsilon} \xi \right)^{-1-1/\gamma} \mathbf{1}_{\xi < 0} d\tau dx \leq C \int_{\mathbb{R}} W_0^2 dx,$$

and for the second integral

$$\int_{\mathbb{R}} W_0^2 \int_0^t \frac{1}{\phi_\epsilon} \mathbf{1}_{\xi > 0} d\tau dx \leq \frac{C}{\epsilon} \int_{\mathbb{R}} W_0^2 \int_0^{x/s} \mathbf{1}_{\xi > 0} d\tau dx = \frac{C}{\epsilon} \int_{\mathbb{R}} W_0^2 x \mathbf{1}_{x \geq 0} dx.$$

Returning to (6.51), we get

$$\begin{aligned} &\int_{\mathbb{R}} V_2^2(t) dx + \frac{3}{2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x V_2|^2 dx d\tau \\ &\leq \int_{\mathbb{R}} V_2^2(0) dx + C \|W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} W_0\|_{L^2(\mathbb{R}_+)}^2 + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}, \end{aligned}$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

i.e.

$$E_0(t; V_2) + \frac{3}{2} \int_0^t D_0(\tau; V_2) d\tau \leq E_0(0; V_2) + C \|W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} W_0\|_{L^2(\mathbb{R}_+)}^2 + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \quad (6.55)$$

Remark 6.8. *Note that (6.55) does not depend on $T > 0$. This is due to the assumption that $\sqrt{x} W_0 \in L^2(\mathbb{R}_+)$. If this assumption is removed, one can still obtain a bound, but it is not global in time anymore. Indeed, we can simply apply Holder and Young's inequality to get*

$$\left| \int_0^t \int_{\mathbb{R}} W_0 \partial_x V_2 \right| \leq \|\phi_\epsilon^{-1/2}\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V_2\|_{L_{t,x}^2} \|W_0\|_{L_{t,x}^2} \leq \frac{Ct}{\epsilon} \|W_0\|_{L_x^2}^2 + \frac{1}{4} \int_0^t D_0(\tau; V_2) d\tau.$$

Thus instead of (6.55) we obtain

$$E_0(t; V_2) + \frac{3}{2} \int_0^t D_0(\tau; V_2) d\tau \leq E_0(0; V_2) + \frac{Ct}{\epsilon} \|W_0\|_{L_x^2}^2 + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \quad (6.56)$$

3.4.2. Estimates for $k = 1$

From (6.37), we deduce that η_2 solves

$$\partial_t \eta_2 - \partial_x (\phi_\epsilon(v_\epsilon) \partial_x \eta_2) = \mathcal{L}_\epsilon(\eta_2) + S_\epsilon(\partial_x V_1).$$

Multiplying by η_2 and integrating in space and time, we get

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} |\eta_2(t)|^2 dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x \eta_2|^2 dx d\tau - \frac{1}{2} \int_{\mathbb{R}} |\eta_2(0)|^2 dx \\ &= \int_0^t \int_{\mathbb{R}} \eta_2 \frac{\partial_x^2 H(\partial_x V_1)}{v_\epsilon - 1} dx d\tau + \int_0^t \int_{\mathbb{R}} \eta_2 \frac{\partial_x^2 W_0}{v_\epsilon - 1} dx d\tau + \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\eta_2) \eta_2 dx d\tau \\ &=: \sum_{n=1}^3 I_n. \end{aligned} \quad (6.57)$$

We now estimate each of I_1, I_2, I_3 . For I_1 , we first integrate by parts to get

$$I_1 = - \int_0^t \int_{\mathbb{R}} \partial_x \eta_2 \frac{\partial_x H(\partial_x V_1)}{v_\epsilon - 1} + \int_0^t \int_{\mathbb{R}} \eta_2 \frac{v'_\epsilon \partial_x H(\partial_x V_1)}{(v_\epsilon - 1)^2} =: (1a) + (1b).$$

Using (6.31),

$$|(1a)| \leq C \int_0^t \int_{\mathbb{R}} |\partial_x \eta_2| \left| \frac{1}{v_\epsilon - 1} \right| \left(\left| \frac{\partial_x V_1}{v_\epsilon - 1} \right|^2 + \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x V_1| |\partial_x^2 V_1| \right).$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

The first term can be estimated as

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} |\partial_x \eta_2| \left| \frac{1}{v_\epsilon - 1} \right| \left| \frac{\partial_x V_1}{v_\epsilon - 1} \right|^2 &\leq \left\| \frac{1}{\phi_\epsilon (v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \|\eta_1\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \eta_2\|_{L_{t,x}^2} \\ &\leq \frac{C'}{\epsilon^{7/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}, \end{aligned}$$

where we have used (6.48). Note that the identity (6.50) implies that

$$\left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \leq \|\sqrt{\phi_\epsilon} \partial_x \eta_1\|_{L_{t,x}^2} + \|v'_\epsilon (v_\epsilon - 1)^{-2} \sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2} \leq \frac{C'}{\epsilon} \|V_1\|_{\mathcal{X}}. \quad (6.58)$$

Therefore, we have

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+1}} |\partial_x \eta_2| \left| \frac{\partial_x^2 V_1}{v_\epsilon - 1} \right| \left| \frac{\partial_x V_1}{v_\epsilon - 1} \right| \\ \leq \left\| \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+1} \phi_\epsilon} \right\|_{L_{t,x}^\infty} \|\eta_1\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x \eta_2\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \\ \leq \frac{C'}{\epsilon^{7/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \end{aligned}$$

Next note that using (6.31),

$$\begin{aligned} |(1b)| &\leq \int_0^t \int_{\mathbb{R}} \frac{v'_\epsilon}{(v_\epsilon - 1)^3} |\partial_x V_2| |\partial_x H(\partial_x V_1)| \\ &\leq \int_0^t \int_{\mathbb{R}} |\partial_x V_2| \frac{v'_\epsilon}{(v_\epsilon - 1)^3} \left(\left| \frac{\partial_x V_1}{v_\epsilon - 1} \right|^2 + \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x V_1| |\partial_x^2 V_1| \right). \end{aligned}$$

Then estimating in the same way as (1a), we also find that $(1b) \leq (C'/\epsilon^{7/2}) \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}$ and so

$$|I_1| \leq \frac{C'}{\epsilon^{7/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \quad (6.59)$$

Integrating by parts,

$$I_2 = - \int_0^t \int_{\mathbb{R}} \frac{\partial_x W_0}{v_\epsilon - 1} \partial_x \eta_2 + \int_0^t \int_{\mathbb{R}} \frac{v'_\epsilon \partial_x W_0}{(v_\epsilon - 1)^2} \frac{\partial_x V_2}{v_\epsilon - 1}.$$

We first compute using Young's inequality that

$$\left| \int_0^t \int_{\mathbb{R}} \frac{\partial_x W_0}{v_\epsilon - 1} \partial_x \eta_2 \right| \leq \frac{1}{8} \int_0^t D_1(\tau; V_2) d\tau + C \int_0^t \int_{\mathbb{R}} \frac{1}{\phi_\epsilon} \left(\frac{\partial_x W_0}{v_\epsilon - 1} \right)^2.$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

The last integral can be estimated by

$$\int_0^t \int_{\mathbb{R}} \frac{1}{\phi_\epsilon} \left(\frac{\partial_x W_0}{\nu_\epsilon - 1} \right)^2 \leq \int_{\mathbb{R}} (\partial_x W_0)^2 \int_0^t \frac{1}{\phi_\epsilon(\nu_\epsilon - 1)^2} (\mathbf{1}_{\xi < 0} + \mathbf{1}_{\xi > 0}) d\tau dx,$$

with $\xi := x - s\tau$. The bound (6.10) then yields for the part $\{\xi < 0\}$ the estimate

$$\begin{aligned} \int_0^t \frac{1}{\phi_\epsilon(\nu_\epsilon - 1)^2} \mathbf{1}_{\xi < 0} &\leq \frac{C}{\epsilon} \int_0^t \left(B - \frac{A_1}{\epsilon} \xi \right)^{-1+1/\gamma} \mathbf{1}_{\xi < 0} \\ &\leq C \left[\left(B - \frac{A_1}{\epsilon} (x - st) \right)^{1/\gamma} - \left(B - \frac{A_1}{\epsilon} (x - \max(x, 0)) \right)^{1/\gamma} \right] \mathbf{1}_{x-st \leq 0}. \end{aligned}$$

Note that the bound obtained in the end is a positive nondecreasing function that goes to $+\infty$ as t goes to $+\infty$, for every x . The inequality (6.8) shows that this bound is optimal, i.e. that a similar lower bounds holds for this integral. Hence it seems complicated to bound the contribution of the integral I_2 uniformly in time. However, it is always possible to write a bound of the form

$$\int_0^t \frac{1}{\phi_\epsilon(\nu_\epsilon - 1)^2} \mathbf{1}_{\xi < 0} \leq C \left(\frac{t}{\epsilon} \right)^{1/\gamma},$$

uniformly in x . For the part $\{\xi > 0\}$, one has as before

$$\int_{\mathbb{R}} (\partial_x W_0)^2 \int_0^t \frac{1}{\phi_\epsilon(\nu_\epsilon - 1)^2} \mathbf{1}_{\xi > 0} d\tau dx \leq \frac{C}{\epsilon} \int_{\mathbb{R}} (\partial_x W_0)^2 \int_0^{x/s} \mathbf{1}_{\xi > 0} d\tau dx = \frac{C}{\epsilon} \int_{\mathbb{R}} (\partial_x W_0)^2 x \mathbf{1}_{x \geq 0} dx.$$

For the second integral appearing in I_2 , we use the same splitting between $\xi < 0$ and $\xi > 0$, together with (6.10) to obtain again

$$\left| \int_0^t \int_{\mathbb{R}} \frac{\nu'_\epsilon \partial_x W_0}{(\nu_\epsilon - 1)^2} \frac{\partial_x V_2}{\nu_\epsilon - 1} \right| \leq \frac{1}{2\epsilon^2} \int_0^t D_0(\tau, V_2) d\tau + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \|\partial_x W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2.$$

This leads to

$$|I_2| \leq \frac{1}{8} \int_0^t D_1(\tau; V_2) d\tau + \frac{1}{2\epsilon^2} \int_0^t D_0(\tau; V_2) d\tau + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \|\partial_x W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2. \quad (6.60)$$

By comparison with (6.55), we see that multiplying by the singular weight $1/(\nu_\epsilon - 1)$ prevents the estimates from being uniform in time when $W_0 \neq 0$.

Remark 6.9. *If we do not suppose that $\sqrt{x} \partial_x W_0 \in L^2(\mathbb{R}_+)$, but only that $W_0 \in H^1(\mathbb{R})$, it*

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

is still possible to obtain a weaker estimate:

$$\begin{aligned} |I_2| &\leq \left\| \frac{1}{(v_\epsilon - 1)\sqrt{\phi_\epsilon}} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x \eta_2\|_{L_{t,x}^2} \|\partial_x W_0\|_{L_{t,x}^2} + \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^3 \sqrt{\phi_\epsilon}} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V_2\|_{L_{t,x}^2} \|\partial_x W_0\|_{L_{t,x}^2} \\ &\leq \frac{Ct}{\epsilon} \|\partial_x W_0\|_{L_x^2}^2 + \frac{C}{\epsilon^2} \int_0^t D_0(\tau; V_2) d\tau + \frac{1}{8} \int_0^t D_1(\tau; V_2) d\tau. \end{aligned}$$

Finally, we use Equation (6.39) of Lemma 6.5 with $\alpha = 1/8$ in order to bound I_3 :

$$|I_3| = \left| \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\eta_2) \eta_2 \right| \leq \frac{1}{8} \int_0^t \int_{\mathbb{R}} \phi_\epsilon (\partial_x \eta_2)^2 + \frac{C}{\epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon (\partial_x V_2)^2$$

Collecting our estimates for $I_1 - I_3$ and returning to (6.57), we find

$$\begin{aligned} E_1(t; V_2) + \frac{3}{2} \int_0^t D_1(\tau; V_2) d\tau &\leq E_1(0; V_2) + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \|\partial_x W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2 \\ &\quad + \frac{C'}{\epsilon^{7/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} + \frac{C}{\epsilon^2} \int_0^t D_0(\tau; V_2) d\tau, \end{aligned} \tag{6.61}$$

which yields, after multiplication by ϵ^2 ,

$$\begin{aligned} \epsilon^2 E_1(t; V_2) + \frac{3}{2} \epsilon^2 \int_0^t D_1(\tau; V_2) d\tau &\leq \epsilon^2 E_1(0; V_2) + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + C\epsilon \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2 \\ &\quad + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} + C \int_0^t D_0(\tau; V_2) d\tau. \end{aligned} \tag{6.62}$$

Remark 6.10. *If we do not suppose that $\sqrt{x} \partial_x W_0$ belong to $L^2(\mathbb{R}_+)$, we still have*

$$\epsilon^2 E_1(t; V_2) + \frac{3}{2} \epsilon^2 \int_0^t D_1(\tau; V_2) d\tau \leq \epsilon^2 E_1(0; V_2) + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} + Ct\epsilon \|\partial_x W_0\|_{L_{t,x}^2}^2.$$

3.4.3. Estimates for $k = 2$

Equation (6.38) gives that $\partial_x \eta_2$ solves

$$\partial_t(\partial_x \eta_2) - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x^2 \eta_2) = \partial_x S_\epsilon(\partial_x V_1) + \mathcal{L}_\epsilon(\partial_x \eta_2) + \mathcal{C}_\epsilon(\eta_2).$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

Multiplying by $\partial_x \eta_2$ and integrating in space and time leads to

$$\begin{aligned}
& \frac{1}{2} \int_{\mathbb{R}} |\partial_x \eta_2(t)|^2 dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x^2 \eta_2|^2 dx d\tau - \frac{1}{2} \int_{\mathbb{R}} |\partial_x \eta_2(0)|^2 dx \\
&= \int_0^t \int_{\mathbb{R}} \frac{\partial_x^2 H(\partial_x V_1)}{v_\epsilon - 1} \partial_x^2 \eta_2 + \int_0^t \int_{\mathbb{R}} \frac{\partial_x^2 W_0}{v_\epsilon - 1} \partial_x^2 \eta_2 + \int_0^t \mathcal{L}_\epsilon(\partial_x \eta_2) \partial_x \eta_2 + \int_0^t \int_{\mathbb{R}} \partial_x \eta_2 \mathcal{C}_\epsilon(\eta_2) \\
&=: \sum_{n=1}^4 J_n.
\end{aligned} \tag{6.63}$$

We now estimate J_1, J_2, J_3, J_4 . Using (6.32),

$$\begin{aligned}
|J_1| &\leq C \int_0^t \int_{\mathbb{R}} \frac{|\partial_x^2 \eta_2|}{v_\epsilon - 1} \left(\frac{(v_\epsilon - 1)^{\gamma-2}}{\epsilon} |\partial_x V_1|^2 + \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x V_1| |\partial_x^3 V_1| + \frac{\epsilon}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x^2 V_1|^2 \right) \\
&\leq \frac{C}{\epsilon^2} \|\eta_1\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x^2 \eta_2\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2} + C\epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x V_1|}{v_\epsilon - 1} \frac{|\partial_x^2 \eta_2|}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x^3 V_1| \\
&\quad + C\epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x^2 \eta_2|}{v_\epsilon - 1} \frac{|\partial_x^2 V_1|^2}{(v_\epsilon - 1)^{\gamma+2}}.
\end{aligned} \tag{6.64}$$

We have that

$$\begin{aligned}
\epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x V_1|}{v_\epsilon - 1} \frac{|\partial_x^2 \eta_2|}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x^3 V_1| &\leq C \|\eta_1\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} \frac{|\partial_x^3 V_1|}{v_\epsilon - 1} \phi_\epsilon |\partial_x^2 \eta_2| \\
&\leq C \|\eta_1\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \partial_x^2 \eta_2 \right\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 V_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2}.
\end{aligned}$$

We compute that

$$\frac{\partial_x^3 V_1}{v_\epsilon - 1} = \partial_x^2 \eta_1 + \frac{2(v'_\epsilon)}{v_\epsilon - 1} \partial_x \eta_1 + \frac{\partial_x V_1}{v_\epsilon - 1} \left[\frac{(v'_\epsilon)^2}{(v_\epsilon - 1)^2} + \partial_x \left(\frac{v'_\epsilon}{v_\epsilon - 1} \right) \right],$$

hence

$$\left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 V_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \leq \frac{C'}{\epsilon^2} \|V_1\|_{\mathcal{X}}, \tag{6.65}$$

which yields with the previous computations that

$$\epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x V_1|}{v_\epsilon - 1} \frac{|\partial_x^2 \eta_2|}{(v_\epsilon - 1)^{\gamma+2}} |\partial_x^3 V_1| \leq \frac{C'}{\epsilon^{11/2}} \|V_1\|_{\mathcal{X}} \|V_2\|_{\mathcal{X}}^2.$$

To estimate the final integral appearing in (6.64), let us first note that since

$$\partial_x(\sqrt{\phi_\epsilon} \partial_x \eta_1) = \frac{v'_\epsilon \phi'_\epsilon}{2\sqrt{\phi_\epsilon}} \partial_x \eta_1 + \sqrt{\phi_\epsilon} \partial_x^2 \eta_1,$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

we have that

$$\|\partial_x(\sqrt{\phi_\epsilon}\partial_x\eta_1)\|_{L_{t,x}^2} \leq \frac{C}{\epsilon} \|\sqrt{\phi_\epsilon}\partial_x\eta_1\|_{L_{t,x}^2} + \|\sqrt{\phi_\epsilon}\partial_x^2\eta_1\|_{L_{t,x}^2} \leq \frac{C'}{\epsilon^2} \|V_1\|_{\mathcal{X}}. \quad (6.66)$$

Therefore,

$$\|\sqrt{\phi_\epsilon}\partial_x\eta_1\|_{L_t^2 L_x^\infty} \leq \|\sqrt{\phi_\epsilon}\partial_x\eta_1\|_{L_{t,x}^2}^{1/2} \|\partial_x(\sqrt{\phi_\epsilon}\partial_x\eta_1)\|_{L_{t,x}^2}^{1/2} \leq \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}} \quad (6.67)$$

Using this estimate and (6.50), we have that

$$\begin{aligned} & \epsilon \int_0^t \int_{\mathbb{R}} \frac{|\partial_x^2 \eta_2|}{v_\epsilon - 1} \frac{|\partial_x^2 V_1|^2}{(v_\epsilon - 1)^{\gamma+2}} \\ & \leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x^2 \eta_2| \frac{|\partial_x^2 V_1|^2}{(v_\epsilon - 1)^2} \leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x^2 \eta_2| \left(|\partial_x \eta_1|^2 + \frac{(v'_\epsilon)^2 |\partial_x V_1|^2}{(v_\epsilon - 1)^4} \right) \\ & \leq C \|\sqrt{\phi_\epsilon} \partial_x^2 \eta_2\|_{L_{t,x}^2} \left(\|\partial_x \eta_1\|_{L_t^\infty L_x^2} \|\sqrt{\phi_\epsilon} \partial_x \eta_1\|_{L_{t,x}^2} + \frac{C}{\epsilon^2} \|\eta_1\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2} \right) \\ & \leq \frac{C'}{\epsilon^{11/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \end{aligned}$$

In summary, we find $J_1 \leq (C'/\epsilon^{11/2}) \|V_2\|_{\mathcal{X}} \|V_1\|_{\mathcal{X}}^2$.

Remark 6.11. *From the estimate for J_1 we can deduce that*

$$\|\sqrt{\phi_\epsilon} |\partial_x \eta|^2\|_{L_{t,x}^2} \leq \frac{C'}{\epsilon^{5/2}} \|V\|_{\mathcal{X}}, \quad (6.68)$$

$$\|\sqrt{\phi_\epsilon} |\partial_x V|^2\|_{L_{t,x}^2} \leq \frac{C'}{\epsilon} \|V\|_{\mathcal{X}}, \quad (6.69)$$

$$\|\sqrt{\phi_\epsilon} \partial_x V\|_{L_t^2 L_x^\infty} \leq \frac{C'}{\sqrt{\epsilon}} \|V\|_{\mathcal{X}}, \quad (6.70)$$

which will be useful for later estimates.

For J_2 , we proceed as in the $k = 1$ case and obtain an estimate similar to (6.60):

$$|J_2| = \left| \int_0^t \int_{\mathbb{R}} \frac{\partial_x^2 W_0}{v_\epsilon - 1} \partial_x^2 \eta_2 \right| \leq \frac{1}{6} \int_0^t D_2(\tau; V_2) d\tau + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \|\partial_x^2 W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2.$$

Remark 6.12. *If $\sqrt{x} \partial_x^2 W_0 \notin L^2(\mathbb{R}_+)$, we still have using the Holder and Young inequalities that*

$$|J_2| \leq \frac{C}{\sqrt{\epsilon}} \|\partial_x^2 W_0\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x^2 \eta_2\|_{L_{t,x}^2} \leq \frac{Ct}{\epsilon} \|\partial_x^2 W_0\|_{L_x^2}^2 + \frac{1}{6} \int_0^t D_2(\tau; V_2) d\tau.$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

We now use Equation (6.39) of Lemma 6.5 with $\alpha = 1/6$ to obtain that

$$|J_3| = \left| \int_0^t \mathcal{L}_\epsilon(\partial_x \eta_2) \partial_x \eta_2 \right| \leq \frac{1}{6} \int_0^t \int_{\mathbb{R}} \phi_\epsilon (\partial_x^2 \eta_2)^2 + \frac{C}{\epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon (\partial_x \eta_2)^2.$$

It now remains to estimate the term involving $\mathcal{C}_\epsilon$, i.e. J_4 . We use this time Equation (6.40) of Lemma 6.5 with $\alpha = 1/6$ to get

$$|J_4| = \left| \int_0^t \int_{\mathbb{R}} \partial_x \eta_2 \mathcal{C}_\epsilon(\eta_2) \right| \leq \frac{1}{6} \int_0^t \int_{\mathbb{R}} \phi_\epsilon (\partial_x^2 \eta)^2 + \frac{C}{\epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon \left((\partial_x \eta_2)^2 + \frac{(\partial_x V_2^2)}{\epsilon^2} \right).$$

Returning to (6.63) with our estimates for $K_1 - K_7$ and $J_1 - J_3$, we have

$$\begin{aligned} & E_2(t; V_2) + \int_0^t D_2(\tau; V_2) d\tau \\ & \leq E_2(0; V_2) + C \left(\frac{t}{\epsilon} \right)^{1/\gamma} \|\partial_x^2 W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2 + \frac{C'}{\epsilon^{11/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} \\ & \quad + \frac{C}{\epsilon^2} \int_0^t D_1(\tau; V_2) d\tau + \frac{C}{\epsilon^4} \int_0^t D_0(\tau; V_2) d\tau. \end{aligned}$$

We may assume that each $C > 0$ appearing on the right hand-side is bounded by $B_0 = B_0(\gamma, \nu_+, s) > 0$. Thus letting $k_0 := 1/(2B_0)$ and multiplying by $k_0 \epsilon^4$ results in

$$\begin{aligned} & k_0 \epsilon^4 E_2(t; V_2) + k_0 \epsilon^4 \int_0^t D_2(\tau; V_2) d\tau \\ & \leq k_0 \epsilon^4 E_2(0; V_2) + \left(\frac{t}{\epsilon} \right)^{1/\gamma} \epsilon^4 \|\partial_x^2 W_0\|_{L_x^2}^2 + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} \\ & \quad + \frac{\epsilon^2}{2} \int_0^t D_1(\tau; V_2) d\tau + \frac{1}{2} \int_0^t D_0(\tau; V_2) d\tau + \epsilon^3 \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2. \end{aligned} \tag{6.71}$$

Adding to what we found for $k = 1$, i.e. (6.62), we find after simplifying the terms containing D_1 that

$$\begin{aligned} & \epsilon^2 E_1(t; V_2) + \epsilon^2 \int_0^t D_1(\tau; V_2) d\tau + k_0 \epsilon^4 E_2(t; V_2) + k_0 \epsilon^4 \int_0^t D_2(\tau; V_2) d\tau \\ & \leq \epsilon^2 E_1(0; V_2) + k_0 \epsilon^4 E_2(0; V_2) + \left(\frac{t}{\epsilon} \right)^{1/\gamma} \left(C \epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} \\ & \quad + C \epsilon \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2 + \epsilon^3 \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2 + \left(C + \frac{1}{2} \right) \int_0^t D_0(\tau; V_2) d\tau. \end{aligned}$$

Again, we may assume that each $C + 1/2, C > 0$ that appear on the right-hand side is bounded by $B_1 = B_1(\gamma, \nu_+, s) > 0$. Without loss of generality, we may assume that

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

$B_1 > 1$. Thus letting $k_1 := 1/(2B_1)$ and multiplying by k_1 yields

$$\begin{aligned} & k_1 \epsilon^2 E_1(t; V_2) + k_1 \epsilon^2 \int_0^t D_1(\tau; V_2) d\tau + k_1 k_0 \epsilon^4 E_2(t; V_2) + k_1 k_0 \epsilon^4 \int_0^t D_2(\tau; V_2) d\tau \\ & \leq k_1 \epsilon^2 E_1(0; V_2) + k_1 k_0 \epsilon^4 E_2(0; V_2) + \left(\frac{t}{\epsilon}\right)^{1/\gamma} \left(\epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} \\ & \quad + \epsilon \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2 + \epsilon^3 \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2 + \frac{1}{2} \int_0^t D_0(\tau; V_2) d\tau. \end{aligned}$$

Adding this inequality to the one that we obtained for $k = 0$, i.e. (6.55), and simplifying the terms depending on D_0 , we get

$$\begin{aligned} & E_0(t; V_2) + \int_0^t D_0(\tau; V_2) d\tau + k_1 \epsilon^2 \left(E_1(t; V_2) + \int_0^t D_1(\tau; V_2) d\tau \right) \\ & + k_1 k_0 \epsilon^4 \left(E_2(t; V_2) + \int_0^t D_2(\tau; V_2) d\tau \right) \\ & \leq E_0(0; V_2) + k_1 \epsilon^2 E_1(0; V_2) + k_1 k_0 \epsilon^4 E_2(0; V_2) + \left(\frac{t}{\epsilon}\right)^{1/\gamma} \left(\epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) \\ & + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} + \epsilon \|\sqrt{x} \partial_x W_0\|_{L^2(\mathbb{R}_+)}^2 + \epsilon^4 \|\sqrt{x} \partial_x^2 W_0\|_{L^2(\mathbb{R}_+)}^2 \\ & + C \|W_0\|_{L_x^2}^2 + \frac{C}{\epsilon} \|\sqrt{x} W_0\|_{L^2(\mathbb{R}_+)}^2. \end{aligned}$$

Again, we may assume that each $C > 0$ in the right-hand side is bounded by $B_2 = B_2(s, \gamma, \nu_+) > 1$. We set $k_2 := 1/B_2$ and multiply by k_2 . In order to simplify computations, we also define

$$c_0 := k_2, \quad c_1 := k_2 k_1, \quad c_2 := k_2 k_1 k_0. \quad (6.72)$$

We then obtain

$$\begin{aligned} & \sum_{k=0}^2 c_k \epsilon^{2k} \left(E_k(t; V_2) + \int_0^t D_k(\tau; V_2) d\tau \right) \\ & \leq \sum_{k=0}^2 \left(c_k \epsilon^{2k} E_k(0; V_2) + \epsilon^{2k-1} \|\sqrt{x} \partial_x^k W_0\|_{L^2(\mathbb{R}_+)}^2 \right) + \|W_0\|_{L_x^2}^2 \\ & \quad + \left(\frac{t}{\epsilon}\right)^{1/\gamma} \left(\epsilon^2 \|\partial_x W_0\|_{L_x^2}^2 + \epsilon^4 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}. \end{aligned}$$

We use the inequality $t^{1/\gamma} \leq T^{1/\gamma}$ in the right-hand side, and then take the supremum

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

in time in the left-hand side to obtain

$$\begin{aligned}
& \sup_{t \in [0, T]} \left(\sum_{k=0}^2 c_k \epsilon^{2k} \left[E_k(t; V_2) + \int_0^t D_k(\tau; V_2) d\tau \right] \right) \\
& \leq \sum_{k=0}^2 \left[c_k \epsilon^{2k} E_k(0, V_2) + \epsilon^{2k-1} \|\sqrt{x} \partial_x^k W_0\|_{L^2(\mathbb{R}_+)}^2 \right] \\
& + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} + \|W_0\|_{L_x^2}^2 + \epsilon^2 \left(\frac{T}{\epsilon} \right)^{1/\gamma} \left(\|\partial_x W_0\|_{L_{t,x}^2}^2 + \epsilon^2 \|\partial_x^2 W_0\|_{L_x^2}^2 \right).
\end{aligned} \tag{6.73}$$

Note that, when $W_0 = 0$, one can take $T = +\infty$ and obtain a global bound. Using the constants c_0, c_1, c_2 in the definition of the norm $\|\cdot\|_{\mathcal{X}}$, (6.73) can be recast as

$$\begin{aligned}
\|V_2\|_{\mathcal{X}}^2 & \leq \sum_{k=0}^2 \left[c_k \epsilon^{2k} E_k(0, V_2) + \epsilon^{2k-1} \|\sqrt{x} \partial_x^k W_0\|_{L^2(\mathbb{R}_+)}^2 \right] + \|W_0\|_{L_x^2}^2 \\
& + \epsilon^2 \left(\frac{T}{\epsilon} \right)^{1/\gamma} \left(\|\partial_x W_0\|_{L_{t,x}^2}^2 + \epsilon^2 \|\partial_x^2 W_0\|_{L_x^2}^2 \right) + \frac{C'}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}} \\
& \leq C' \delta^2 \epsilon^{3/2} \|V_2\|_{\mathcal{X}} + \delta_0^2 \epsilon^3,
\end{aligned}$$

where we have used our assumption on the initial data (6.45). Taking $\delta < 1$ to be such that $\delta < 1/(\sqrt{2}C')$, defining $\delta_0 := \delta/2$ and using Young's inequality gives us

$$\|V_2\|_{\mathcal{X}}^2 \leq \delta^2 \epsilon^3, \tag{6.74}$$

and so $V_2 \in B_\delta$ as required. This completes the first part of Proposition 6.2.

Remark 6.13. *When we do not suppose that $\sqrt{x} \partial_x^k W_0 \in L^2(\mathbb{R}_+)$ for $k = 0, 1, 2$, we still obtain that*

$$\|V_2\|_{\mathcal{X}}^2 \leq \sum_{k=0}^2 \left[c_k \epsilon^{2k} E_k(0; V_2) + T \epsilon^{2k-1} \|\partial_x^k W_0\|_{L_x^2}^2 \right] + C' \delta^2 \epsilon^{3/2} \|V_2\|_{\mathcal{X}}.$$

Hence the proof of the proposition still works after suitable modification of the assumption on the initial data. In this case, one has to impose the condition

$$\sum_{k=0}^2 \left[c_k \epsilon^{2k} E_k(0; V_2) + T \epsilon^{2k-1} \|\partial_x^k W_0\|_{L_x^2}^2 \right] \leq \delta_0^2 \epsilon^3.$$

3.5. The map $\mathcal{A}^\epsilon$ is a contraction

In this subsection we aim to show that the map $\mathcal{A}^\epsilon$ is a contraction. We consider two elements $V_1, V'_1 \in B_\delta$ and define $V_2 := \mathcal{A}^\epsilon(V_1)$ and $V'_2 := \mathcal{A}^\epsilon(V'_1)$. Our goal is to show that there exists $\delta < 1$ with $\|V_2 - V'_2\|_{\mathcal{X}} \leq \delta \|V_1 - V'_1\|_{\mathcal{X}}$. Defining $\tilde{V}_1 := V_1 - V'_1$ and

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

$\tilde{V}_2 := V_2 - V_1'$, the following equations are satisfied:

$$\begin{cases} \partial_t \tilde{V}_2 - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x \tilde{V}_2) = \partial_x (H(\partial_x V_1) - H(\partial_x V_1')), \\ \partial_t \tilde{\eta}_2 - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x \tilde{\eta}_2) = \frac{\partial_x^2 (H(\partial_x V_1) - H(\partial_x V_1'))}{v_\epsilon - 1} + \mathcal{L}_\epsilon(\tilde{\eta}_2), \\ \partial_t \partial_x \tilde{\eta}_2 - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x^2 \tilde{\eta}_2) = \partial_x \left(\frac{\partial_x^2 (H(\partial_x V_1) - H(\partial_x V_1'))}{v_\epsilon - 1} \right) + \mathcal{L}_\epsilon(\partial_x \tilde{\eta}_2) + \mathcal{C}_\epsilon(\tilde{\eta}_2). \end{cases} \quad (6.75)$$

We see that $\tilde{V}_2$, $\tilde{\eta}_2$ and $\partial_x \tilde{\eta}_2$ solve the same equations than V_2 , η_2 , $\partial_x \eta_2$ respectively, up to two differences:

- The constant term W_0 is equal to zero here, as well as the initial data,
- The nonlinear terms (depending on H) are different.

With this in mind, multiplying the equations by $\tilde{V}_2$, $\tilde{\eta}_2$, $\partial_x \tilde{\eta}_2$ respectively and using again Lemma 6.5 (taking advantage of the linearity of $\mathcal{L}_\epsilon$ and $\mathcal{C}_\epsilon$), we see that the computations of the previous section can be repeated. We get a similar estimate after suitable modification of the nonlinear terms. Since here W_0 and the initial data are equal to zero, we are left with

$$\begin{aligned} \|\tilde{V}_2\|_{\mathcal{X}}^2 &\leq C \left| \int_0^t \int_{\mathbb{R}} \tilde{V}_2 \partial_x (H(\partial_x V_1) - H(\partial_x V_1')) \right| + C\epsilon^2 \left| \int_0^t \int_{\mathbb{R}} \tilde{\eta}_2 \left(\frac{\partial_x^2 (H(\partial_x V_1) - H(\partial_x V_1'))}{v_\epsilon - 1} \right) \right| \\ &\quad + C\epsilon^4 \left| \int_0^t \int_{\mathbb{R}} \partial_x \tilde{\eta}_2 \partial_x \left(\frac{\partial_x^2 (H(\partial_x V_1) - H(\partial_x V_1'))}{v_\epsilon - 1} \right) \right| =: C \sum_{k=1}^3 \epsilon^{2k-2} |L_k|. \end{aligned} \quad (6.76)$$

To estimate each L_k we will need the following result.

Lemma 6.6. *If $f_1, f_2 \in \mathcal{X}$ are such that $\| \frac{f_1}{v_\epsilon - 1} \|_\infty + \| \frac{f_2}{v_\epsilon - 1} \|_\infty \leq \delta$ for some $\delta < 1$, then there*

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

exists a constant $C > 0$ independent of ϵ such that

$$|H(f_1) - H(f_2)| \leq \frac{C\phi_\epsilon(v_\epsilon)}{v_\epsilon - 1} |f_1 - f_2| (|f_1| + |f_2|), \quad (6.77)$$

$$\begin{aligned} & |\partial_x H(f_1) - \partial_x H(f_2)| \\ & \leq \frac{C\phi_\epsilon(v_\epsilon)}{v_\epsilon - 1} \left(\frac{v'_\epsilon}{v_\epsilon - 1} |f_1 - f_2| (|f_1| + |f_2|) + |\partial_x(f_1 - f_2)| |f_1| + |\partial_x f_2| |f_1 - f_2| \right), \end{aligned} \quad (6.78)$$

$$\begin{aligned} |\partial_x^2 H(f_1) - \partial_x^2 H(f_2)| & \leq \frac{C(v_\epsilon - 1)^{r-2}}{\epsilon} |f_1 - f_2| (|f_1| + |f_2|) \\ & \quad + \frac{C}{(v_\epsilon - 1)^2} (|\partial_x(f_1 - f_2)| |f_2| + |f_1 - f_2| |\partial_x f_1|) \\ & \quad + \frac{C\phi_\epsilon(v_\epsilon)}{v_\epsilon - 1} |\partial_x(f_1 - f_2)| (|\partial_x f_1| + |\partial_x f_2|) \\ & \quad + \frac{C\phi_\epsilon(v_\epsilon)}{v_\epsilon - 1} \left(|f_1 - f_2| |\partial_x^2 f_1| + |\partial_x^2(f_1 - f_2)| |f_2| + \frac{(\partial_x f_2)^2}{v_\epsilon - 1} |f_1 - f_2| \right). \end{aligned} \quad (6.79)$$

The proof follows the same approach as that of Lemma 6.4, and a very similar result can be seen in Lemma 3.4 of Dalibard et al. 2019. Thus we omit the proof. Integrating by parts and using (6.77),

$$\begin{aligned} |L_1| & = \left| \int_0^t \int_{\mathbb{R}} \partial_x \tilde{V}_2 (H(\partial_x V_1) - H(\partial_x V'_1)) \right| \leq \int_0^t \int_{\mathbb{R}} |\partial_x \tilde{V}_2| \left(\frac{\phi_\epsilon}{v_\epsilon - 1} |\partial_x \tilde{V}_1| (|\partial_x V_1| + |\partial_x V'_1|) \right) \\ & \leq C \int_0^t \int_{\mathbb{R}} \sqrt{\phi_\epsilon} |\partial_x \tilde{V}_2| \frac{|\partial_x \tilde{V}_1|}{v_\epsilon - 1} \left(\sqrt{\phi_\epsilon} |\partial_x V_1| + \sqrt{\phi_\epsilon} |\partial_x V'_1| \right) \\ & \leq \|\tilde{\eta}_1\|_{L_{t,x}^\infty} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}). \end{aligned}$$

Using the definition of the norm,

$$|L_1| \leq \frac{C}{\epsilon^{3/2}} \|\tilde{V}_2\|_{\mathcal{X}} \|\tilde{V}_1\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}). \quad (6.80)$$

To deal with L_2 , we first integrate by parts to get

$$\begin{aligned} L_2 & = - \int_0^t \int_{\mathbb{R}} \tilde{\eta}_2 \frac{v'_\epsilon (\partial_x H(\partial_x V_1) - \partial_x H(\partial_x V'_1))}{(v_\epsilon - 1)^2} - \int_0^t \int_{\mathbb{R}} \frac{\partial_x \tilde{\eta}_2}{v_\epsilon - 1} (\partial_x H(\partial_x V_1) - \partial_x H(\partial_x V'_1)) \\ & =: (2a) + (2b). \end{aligned} \quad (6.81)$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

Using (6.78),

$$\begin{aligned}
|(2a)| &\leq C \int_0^t \int_{\mathbb{R}} |\tilde{\eta}_2| \frac{(v'_\epsilon)^2 \phi_\epsilon}{(v_\epsilon - 1)^4} |\partial_x \tilde{V}_1| (|\partial_x V_1| + |\partial_x V'_1|) + C \int_0^t \int_{\mathbb{R}} |\tilde{\eta}_2| \frac{v'_\epsilon \phi_\epsilon}{(v_\epsilon - 1)^3} |\partial_x^2 \tilde{V}_1| |\partial_x V_1| \\
&\quad + C \int_0^t \int_{\mathbb{R}} |\tilde{\eta}_2| \frac{v'_\epsilon \phi_\epsilon}{(v_\epsilon - 1)^3} |\partial_x^2 V'_1| |\partial_x \tilde{V}_1| \\
&\leq C \left\| \frac{(v'_\epsilon)^2}{(v_\epsilon - 1)^4} \right\|_{L_{t,x}^\infty} \|\tilde{\eta}_2\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x \tilde{V}_1\|_{L_{t,x}^2} (\|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2} + \|\sqrt{\phi_\epsilon} \partial_x V'_1\|_{L_{t,x}^2}) \\
&\quad + C \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \|\tilde{\eta}_2\|_{L_{t,x}^\infty} \times \\
&\quad \left(\left\| \frac{\sqrt{\phi_\epsilon} \partial_x^2 \tilde{V}_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x V_1\|_{L_{t,x}^2} + \left\| \frac{\sqrt{\phi_\epsilon} \partial_x^2 V'_1}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \tilde{V}_1\|_{L_{t,x}^2} \right).
\end{aligned}$$

Using (6.48) and (6.58), we therefore find

$$|(2a)| \leq \frac{C}{\epsilon^{7/2}} \|\tilde{V}_2\|_{\mathcal{X}} \|\tilde{V}_1\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}).$$

For (2b) we use (6.78) to get

$$\begin{aligned}
|(2b)| &\leq \int_0^t \int_{\mathbb{R}} |\partial_x \tilde{\eta}_2| \frac{\phi_\epsilon}{(v_\epsilon - 1)^2} \left(\frac{v'_\epsilon}{v_\epsilon - 1} |\partial_x \tilde{V}_1| (|\partial_x V_1| + |\partial_x V'_1|) + |\partial_x^2 \tilde{V}_1| |\partial_x V_1| + |\partial_x^2 V'_1| |\partial_x \tilde{V}_1| \right) \\
&=: b_1 + b_2 + b_3.
\end{aligned}$$

We then have that

$$\begin{aligned}
b_1 &\leq C \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} |\sqrt{\phi_\epsilon} \partial_x \tilde{\eta}_2| |\sqrt{\phi_\epsilon} \partial_x \tilde{V}_1| (|\eta_1| + |\eta'_1|) \\
&\leq \frac{C}{\epsilon} \|\sqrt{\phi_\epsilon} \partial_x \tilde{\eta}_2\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \tilde{V}_1\|_{L_{t,x}^2} (\|\eta_1\|_{L_{t,x}^\infty} + \|\eta'_1\|_{L_{t,x}^\infty}) \\
&\leq \frac{C}{\epsilon^{7/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}).
\end{aligned}$$

Next,

$$\begin{aligned}
b_2 &\leq \int_0^t \int_{\mathbb{R}} |\sqrt{\phi_\epsilon} \partial_x \tilde{\eta}_2| |\sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{v_\epsilon - 1}| |\eta_1| \leq C \|\eta_1\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x \tilde{\eta}_2\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{v_\epsilon - 1}\|_{L_{t,x}^2} \\
&\leq \frac{C}{\epsilon^{7/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} \|V_1\|_{\mathcal{X}}.
\end{aligned}$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

We find similarly that

$$b_3 \leq \frac{C}{\epsilon^{7/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} \|V_1'\|_{\mathcal{X}}, \quad (6.82)$$

which yields with the previous estimates that

$$\epsilon^2 |L_2| \leq \epsilon^2 \frac{C}{\epsilon^{7/2}} \|\tilde{V}_2\|_{\mathcal{X}} \|\tilde{V}_1\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V_1'\|_{\mathcal{X}}) = \frac{C}{\epsilon^{3/2}} \|\tilde{V}_2\|_{\mathcal{X}} \|\tilde{V}_1\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V_1'\|_{\mathcal{X}}).$$

We now estimate the term L_3 . Using (6.79) and (6.46),

$$\begin{aligned} |L_3| &\leq \frac{C}{\epsilon} \int_0^t \int_{\mathbb{R}} |\partial_x^2 \tilde{\eta}_2| (\nu_\epsilon - 1)^{\gamma-3} |\partial_x \tilde{V}_1| (|\partial_x V_1| + |\partial_x V_1'|) \\ &\quad + C \int_0^t \int_{\mathbb{R}} \frac{|\partial_x^2 \tilde{\eta}_2|}{(\nu_\epsilon - 1)^3} (|\partial_x^2 \tilde{V}_1| |\partial_x V_1| + |\partial_x \tilde{V}_1| |\partial_x^2 V_1|) \\ &\quad + C \int_0^t \int_{\mathbb{R}} |\partial_x^2 \tilde{\eta}_2| \frac{\phi_\epsilon}{(\nu_\epsilon - 1)^2} |\partial_x^2 \tilde{V}_1| (|\partial_x^2 V_1| + |\partial_x^2 V_1'|) \\ &\quad + C \int_0^t \int_{\mathbb{R}} |\partial_x^2 \tilde{\eta}_2| \frac{\phi_\epsilon}{(\nu_\epsilon - 1)^2} (|\partial_x \tilde{V}_1| |\partial_x^3 V_1| + |\partial_x^3 \tilde{V}_1| |\partial_x V_1'|) \\ &\quad + C \int_0^t \int_{\mathbb{R}} |\partial_x^2 \tilde{\eta}_2| \frac{\phi_\epsilon}{(\nu_\epsilon - 1)^3} |\partial_x^2 V_1|^2 |\partial_x \tilde{V}_1| =: \sum_{n=1}^5 M_n. \end{aligned} \quad (6.83)$$

Using (6.58),

$$\begin{aligned} M_1 &\leq \frac{C}{\epsilon} \left\| \frac{(\nu_\epsilon - 1)^{\gamma-2}}{\phi_\epsilon} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \tilde{V}_1\|_{L_{t,x}^2} (\|\eta_1\|_{L_{t,x}^\infty} + \|\eta_1'\|_{L_{t,x}^\infty}) \\ &\leq \frac{C}{\epsilon^{11/2}} \|\tilde{V}_2\|_{\mathcal{X}} \|\tilde{V}_1\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V_1'\|_{\mathcal{X}}). \end{aligned}$$

Similarly,

$$\begin{aligned} M_2 &\leq C \left\| \frac{1}{\phi_\epsilon (\nu_\epsilon - 1)} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2\|_{L_{t,x}^2} \left(\|\sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{\nu_\epsilon - 1}\|_{L_{t,x}^2} \|\eta_1\|_{L_{t,x}^\infty} + \|\tilde{\eta}_1\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \frac{\partial_x^2 V_1}{\nu_\epsilon - 1}\|_{L_{t,x}^2} \right) \\ &\leq \frac{C}{\epsilon^{11/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V_1'\|_{\mathcal{X}}). \end{aligned}$$

Next,

$$\begin{aligned} M_3 &\leq C \int_0^t \int_{\mathbb{R}} |\sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2| \left| \sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{\nu_\epsilon - 1} \right| \left(\left| \frac{\partial_x^2 V_1}{\nu_\epsilon - 1} \right| + \left| \frac{\partial_x^2 V_1'}{\nu_\epsilon - 1} \right| \right) \\ &\leq C \left\| \sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2 \right\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{\nu_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \left(\left\| \frac{\partial_x^2 V_1}{\nu_\epsilon - 1} \right\|_{L_t^\infty L_x^2} + \left\| \frac{\partial_x^2 V_1'}{\nu_\epsilon - 1} \right\|_{L_t^\infty L_x^2} \right). \end{aligned}$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 3. Well-posedness for the integrated system*

Using (6.50), (6.67) and (6.70),

$$\left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 \tilde{V}_1}{\nu_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \leq \left\| \sqrt{\phi_\epsilon} \partial_x \tilde{\eta}_1 \right\|_{L_t^2 L_x^\infty} + \left\| \frac{\nu'_\epsilon}{(\nu_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \partial_x \tilde{V}_1 \right\|_{L_t^2 L_x^\infty} \leq \frac{C}{\epsilon^{3/2}} \|\tilde{V}_1\|_{\mathcal{X}}. \quad (6.84)$$

On the other hand (6.50) also gives us

$$\left\| \frac{\partial_x^2 V_1}{\nu_\epsilon - 1} \right\|_{L_t^\infty L_x^2} \leq C \|\partial_x \eta_1\|_{L_t^\infty L_x^2} + \left\| \frac{\nu'_\epsilon}{(\nu_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \|\eta_1\|_{L_t^\infty L_x^2} \leq \frac{C}{\epsilon^2} \|V_1\|_{\mathcal{X}}, \quad (6.85)$$

and so

$$M_3 \leq \frac{C}{\epsilon^{11/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}).$$

For M_4 , we use again (6.65), thus,

$$\begin{aligned} M_4 &\leq C \left\| \sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2 \right\|_{L_{t,x}^2} \|\tilde{\eta}_1\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 V_1}{\nu_\epsilon - 1} \right\|_{L_{t,x}^2} + C \left\| \sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2 \right\|_{L_{t,x}^2} \|\eta'_1\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 \tilde{V}_1}{\nu_\epsilon - 1} \right\|_{L_{t,x}^2} \\ &\leq \frac{C}{\epsilon^{11/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}). \end{aligned}$$

Lastly, using (6.58) and (6.85),

$$\begin{aligned} M_5 &\leq C \|\tilde{\eta}_1\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \partial_x^2 \tilde{\eta}_2 \right\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V'_1}{\nu_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \left\| \frac{\partial_x^2 V'_1}{\nu_\epsilon - 1} \right\|_{L_t^\infty L_x^2} \\ &\leq \frac{C}{\epsilon^7} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} \|V'_1\|_{\mathcal{X}}^2 = \frac{C}{\epsilon^{11/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} \|V'_1\|_{\mathcal{X}} \left(\frac{\|V'_1\|_{\mathcal{X}}}{\epsilon^{3/2}} \right). \end{aligned}$$

Finally,

$$\begin{aligned} \epsilon^4 |L_3| &\leq \epsilon^4 \frac{C}{\epsilon^{11/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}) \left(1 + \frac{\|V'_1\|_{\mathcal{X}}}{\epsilon^{3/2}} \right) \\ &\leq \frac{C}{\epsilon^{3/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}) \left(1 + \frac{\|V'_1\|_{\mathcal{X}}}{\epsilon^{3/2}} \right) \end{aligned}$$

Gathering our estimates for L_1, L_2, L_3 and returning to (6.76), using the fact that $V_1, V'_1 \in B_\delta$, we get

$$\begin{aligned} \|\tilde{V}_2\|_{\mathcal{X}}^2 &\leq \frac{C}{\epsilon^{3/2}} \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}} (\|V_1\|_{\mathcal{X}} + \|V'_1\|_{\mathcal{X}}) \left(1 + \frac{\|V'_1\|_{\mathcal{X}}}{\epsilon^{3/2}} \right) \\ &\leq \delta C \|\tilde{V}_1\|_{\mathcal{X}} \|\tilde{V}_2\|_{\mathcal{X}}. \end{aligned} \quad (6.86)$$

Therefore taking $\delta < 1/C$ we obtain that $\mathcal{A}^\epsilon$ is a contraction, as required. This concludes the proof of Proposition 6.2.

4. Asymptotic stability of (u_ϵ, v_ϵ)

So far, we have proved that if (u, v) is a solution to the original system such that $v - v_\epsilon, u - u_\epsilon \in L_0^1(\mathbb{R})$ for all positive times then the system can be re-expressed in terms of the integrated quantities (V, W_0) , as (6.25). Furthermore, provided the initial energy is small enough (i.e. satisfying (6.45)) then there exists a unique strong solution to (6.25). In order to prove the existence and uniqueness of strong solutions to the original system (6.1), we will need the following result:

Proposition 6.3. *Provided that*

$$(u - u_\epsilon)(0) \in L_0^1(\mathbb{R}) \cap H^1(\mathbb{R}), \quad \frac{\partial_x(u - u_\epsilon)(0)}{v_\epsilon - 1} \in L^2(\mathbb{R}), \quad (6.87)$$

$$(v - v_\epsilon)(0) \in L_0^1(\mathbb{R}), \quad (6.88)$$

we have

1. $\|u - u_\epsilon\|_{L^\infty(0, t; L^1(\mathbb{R}))} + \|v - v_\epsilon\|_{L^\infty(0, t; L^1(\mathbb{R}))} \leq C(t, \epsilon),$
2. $(u - u_\epsilon)(t), (v - v_\epsilon)(t) \in L_0^1(\mathbb{R})$ for each $t > 0$.

Our strategy of proof is as follows.

- **Step 1:** Establish the stability of the profile u_ϵ . This involves bounding the quantity $u - u_\epsilon$ in $L_t^\infty H_x^1$, which will in particular be needed for the next steps.
- **Step 2:** Use the previous estimates to show that $u - u_\epsilon$ and $v - v_\epsilon$ remain in L^1 for positive times.
- **Step 3:** Complete the remaining part of Proposition 6.3 using the estimates from the previous steps.

For the remainder of this section we define $v := v_\epsilon + \partial_x V$, $w := w_\epsilon + \partial_x W_0$ and $u := w + \phi_\epsilon(v) \partial_x v$. Recall that $\eta = \partial_x V / (v_\epsilon - 1)$ is bounded, with $\|\eta\|_{L_{t,x}^\infty} \leq C\epsilon^{-3/2} \|V\|_{\mathcal{X}} \leq C\delta$ since $V \in B_\delta$, hence $v \geq 1 + (1 - C\delta)(v_\epsilon - 1) > 1$ (up to reducing δ) and $\phi_\epsilon(v)$ is well defined. As a consequence, the pair (w, v) satisfies the system

$$\begin{cases} \partial_t w = 0, \\ \partial_t v - \partial_x w - \partial_x(\phi_\epsilon(v) \partial_x v) = 0. \end{cases}$$

4.1. Stability of the profile $u - u_\epsilon$

From the system solved by (w, v) , we deduce that u is a solution of

$$\partial_t(u - u_\epsilon) - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x(u - u_\epsilon)) = \partial_x[(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x u] =: G. \quad (6.89)$$

We see that the quantity G appearing on the right-hand side depends on the difference $\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)$. In order to control G , we will make use of the following lemma, which is analogous to Lemma 6.4.

6. Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet
de suspension – 4. Asymptotic stability of (u_ϵ, v_ϵ)

Lemma 6.7. *Let $\Delta\phi_\epsilon(f) := \phi_\epsilon(v_\epsilon + f) - \phi_\epsilon(v_\epsilon)$. Then we have*

$$|\Delta\phi_\epsilon(f)| \leq C\phi_\epsilon(v_\epsilon) \frac{|f|}{v_\epsilon - 1}, \quad (6.90)$$

$$|\partial_x \Delta\phi_\epsilon(f)| \leq C\phi_\epsilon(v_\epsilon) \left(\frac{\partial_x v_\epsilon |f|}{(v_\epsilon - 1)^2} + \frac{|\partial_x f|}{v_\epsilon - 1} \right), \quad (6.91)$$

$$|\partial_x^2 \Delta\phi_\epsilon(f)| \leq C\phi_\epsilon(v_\epsilon) \left(\frac{(\partial_x v_\epsilon)^2 |f|}{(v_\epsilon - 1)^3} + \frac{\partial_x v_\epsilon |\partial_x f|}{(v_\epsilon - 1)^2} + \left| \frac{\partial_x f}{v_\epsilon - 1} \right|^2 + \frac{|\partial_x^2 f|}{v_\epsilon - 1} \right). \quad (6.92)$$

As a consequence, we also have that

$$|\phi_\epsilon(v)| = |\phi_\epsilon(v_\epsilon + \partial_x V)| \leq |\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)| + \phi_\epsilon(v_\epsilon) \leq (|\eta| + 1)\phi_\epsilon(v_\epsilon). \quad (6.93)$$

We omit the proof since it follows the same argument as that of Lemma 6.4. We now wish to prove the following existence result.

Lemma 6.8. *Suppose that $(U_0, V_0) \in H^2(\mathbb{R}) \times H^3(\mathbb{R})$ is such that (6.45) is satisfied by (W_0, V_0) , and that (6.87) is also satisfied. Consider the solution $(W_0, V) \in B_\delta \subset \mathcal{X}$ of (6.25) obtained in Proposition 6.2. Then there exists a unique solution $u - u_\epsilon$ to (6.89) such that*

$$u - u_\epsilon \in C([0, T]; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R})). \quad (6.94)$$

Moreover, $u - u_\epsilon$ satisfies

$$\sup_{t \in [0, T]} \left(\sum_{k=0}^1 \left[E_k(t; u - u_\epsilon) + \int_0^t D_k(\tau; u - u_\epsilon) d\tau \right] \right) \leq C, \quad (6.95)$$

where $C = C(\epsilon, \delta, v_+, \gamma, s)$.

Proof. We start from the equation satisfied by $u - u_\epsilon$:

$$\partial_t(u - u_\epsilon) - \partial_x(\phi_\epsilon(v_\epsilon)\partial_x(u - u_\epsilon)) = \partial_x[(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon))\partial_x u] = G.$$

Multiplying by $u - u_\epsilon$ and integrating by parts where appropriate, we have

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}} |u - u_\epsilon|^2(t) dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x(u - u_\epsilon)|^2 dx d\tau - \frac{1}{2} \int_{\mathbb{R}} |u - u_\epsilon|^2(0) dx \\ &= - \int_0^t \int_{\mathbb{R}} \partial_x(u - u_\epsilon)(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x u dx d\tau. \end{aligned}$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension* – 4. *Asymptotic stability of (u_ϵ, v_ϵ)*

With (6.90) and the triangle inequality, the right-hand side can be bounded by

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{R}} \partial_x(u - u_\epsilon)(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x u \right| \\ & \leq C \|\eta\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon)\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x u\|_{L_{t,x}^2} \\ & \leq C \|\eta\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon)\|_{L_{t,x}^2}^2 + C \|\eta\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x u_\epsilon\|_{L_{t,x}^2}^2 \\ & \leq C \delta \|\sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon)\|_{L_{t,x}^2}^2 + C(\delta, v_+, s, \epsilon), \end{aligned}$$

by using $\partial_x u_\epsilon = -s \partial_x v_\epsilon$ and $\|\eta\|_\infty \leq \delta$. Hence we obtain the following estimate (up to reducing δ):

$$\int_{\mathbb{R}} |u - u_\epsilon|^2(t) dx + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x(u - u_\epsilon)|^2 dx \leq \int_{\mathbb{R}} |u - u_\epsilon|^2(0) dx + C(\delta, v_+, s, \epsilon).$$

We now wish to find an estimate for $\partial_x(u - u_\epsilon)$. Notice that (6.89) has the same structure as the equation for V (i.e. (6.33)), up to the source term. As we did for V , we define

$$\mu := \mu(\partial_x u) = \frac{\partial_x(u - u_\epsilon)}{v_\epsilon - 1}. \quad (6.96)$$

The computations that we did for V imply that μ is solution of

$$\partial_t \mu - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x \mu) + \mathcal{L}_\epsilon(\mu) = \frac{\partial_x G}{v_\epsilon - 1}, \quad (6.97)$$

where $\mathcal{L}_\epsilon$ is the same as in Equation (6.37). Multiplying by μ and integrating yields

$$\frac{1}{2} \int_{\mathbb{R}} \mu^2(t) + \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) (\partial_x \mu)^2 = \frac{1}{2} \int_{\mathbb{R}} \mu^2|_{t=0} - \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\mu) \mu + \int_0^t \int_{\mathbb{R}} \mu \frac{\partial_x G}{v_\epsilon - 1}. \quad (6.98)$$

By (6.39), for $\alpha = 1/2$, there exists $C > 0$ such that

$$\left| \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\mu) \mu \right| \leq \frac{1}{2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) (\partial_x \mu)^2 + C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) [\partial_x(u - u_\epsilon)]^2.$$

The first integral can be absorbed by the left-hand side of (6.98). The second integral can be bounded by using the energy estimate satisfied by $u - u_\epsilon$. Hence it is enough to

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension – 4. Asymptotic stability of (u_ϵ, v_ϵ)*

bound the last integral in the right-hand side of (6.98). We compute that

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}} \mu \frac{\partial_x G}{v_\epsilon - 1} \\
&= - \int_0^t \int_{\mathbb{R}} \partial_x \mu \frac{G}{v_\epsilon - 1} + \int_0^t \int_{\mathbb{R}} \mu \frac{\partial_x v_\epsilon G}{(v_\epsilon - 1)^2} \\
&= - \int_0^t \int_{\mathbb{R}} \partial_x \mu \partial_x (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \mu - \int_0^t \int_{\mathbb{R}} \partial_x \mu \partial_x (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \frac{\partial_x u_\epsilon}{v_\epsilon - 1} \\
&\quad - \int_0^t \int_{\mathbb{R}} \partial_x \mu (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \frac{\partial_x^2 (u - u_\epsilon)}{v_\epsilon - 1} - \int_0^t \int_{\mathbb{R}} \partial_x \mu (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \frac{\partial_x^2 u_\epsilon}{v_\epsilon - 1} \\
&\quad + \int_0^t \int_{\mathbb{R}} \frac{\mu \partial_x v_\epsilon}{v_\epsilon - 1} \partial_x (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \mu + \int_0^t \int_{\mathbb{R}} \frac{\mu \partial_x v_\epsilon}{v_\epsilon - 1} \partial_x (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \frac{\partial_x u_\epsilon}{v_\epsilon - 1} \\
&\quad + \int_0^t \int_{\mathbb{R}} \frac{\mu \partial_x v_\epsilon}{(v_\epsilon - 1)^2} (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^2 (u - u_\epsilon) + \int_0^t \int_{\mathbb{R}} \frac{\mu \partial_x v_\epsilon}{(v_\epsilon - 1)^2} (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^2 u_\epsilon \\
&=: \sum_{k=1}^8 I_k.
\end{aligned}$$

By (6.91) and Young's inequality, for any $\alpha > 0$,

$$\begin{aligned}
|I_1| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\mu \partial_x \mu| \left(\frac{\partial_x v_\epsilon |\eta|}{v_\epsilon - 1} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) \\
&\leq C \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2} \left(\frac{1}{\epsilon} \|\sqrt{\phi_\epsilon} \partial_x (u - u_\epsilon)\|_{L_{t,x}^2} \|\eta\|_{L_{t,x}^\infty} + \|\mu\|_{L_t^\infty L_x^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \right) \\
&\leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + \frac{C\delta^2}{\alpha} \left(\frac{1}{\epsilon^2} \|\sqrt{\phi_\epsilon} \partial_x (u - u_\epsilon)\|_{L_{t,x}^2}^2 + \|\mu\|_{L_t^\infty L_x^2}^2 \right),
\end{aligned}$$

where the last line uses the fact that

$$\|\eta\|_{L^\infty} + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \leq \frac{C}{\epsilon^{3/2}} \delta \epsilon^{3/2} \leq C\delta.$$

(see Equation (6.84)). Similarly,

$$\begin{aligned}
|I_2| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) |\partial_x \mu| \left(\frac{\partial_x v_\epsilon |\eta|}{v_\epsilon - 1} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) \frac{\partial_x u_\epsilon}{v_\epsilon - 1} \\
&\leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + \frac{C}{\alpha} \left(\|\sqrt{\phi_\epsilon} \partial_x V\|_{L_{t,x}^2}^2 \left\| \frac{\partial_x v_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty}^2 + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L_{t,x}^2}^2 \right) \left\| \frac{\partial_x u_\epsilon}{v_\epsilon - 1} \right\|_{L_{t,x}^\infty}^2 \\
&\leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + C(\alpha, \epsilon, \delta, s, v_+, \gamma).
\end{aligned}$$

6. Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet
de suspension – 4. Asymptotic stability of (u_ϵ, v_ϵ)

By (6.90),

$$\begin{aligned} |I_3| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x \mu| |\eta| \frac{|\partial_x^2(u - u_\epsilon)|}{v_\epsilon - 1} \\ &\leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + \frac{C\delta^2}{\alpha} \left(\|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + \frac{1}{\epsilon^2} \|\sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon)\|_{L_{t,x}^2}^2 \right) \end{aligned}$$

by using

$$\frac{\partial_x^2(u - u_\epsilon)}{v_\epsilon - 1} = \partial_x \mu + \frac{\partial_x v_\epsilon \partial_x(u - u_\epsilon)}{(v_\epsilon - 1)^2}.$$

Then,

$$\begin{aligned} |I_4| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon |\partial_x \mu| |\eta| \frac{\partial_x^2 u_\epsilon}{v_\epsilon - 1} \leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + \frac{C\delta^2}{\alpha} \|\sqrt{\phi_\epsilon} \frac{\partial_x^2 u_\epsilon}{v_\epsilon - 1}\|_{L_{t,x}^2}^2 \\ &\leq \alpha \|\sqrt{\phi_\epsilon} \partial_x \mu\|_{L_{t,x}^2}^2 + C(\alpha, \epsilon, \delta, s, v_+, \gamma), \end{aligned}$$

$$\begin{aligned} |I_5| &\leq C \int_0^t \int_{\mathbb{R}} \mu^2 \phi_\epsilon \frac{\partial_x v_\epsilon}{v_\epsilon - 1} \left(\frac{\partial_x v_\epsilon |\eta|}{v_\epsilon - 1} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) \\ &\leq \frac{C\delta}{\epsilon^2} \int_0^t \int_{\mathbb{R}} \phi_\epsilon [\partial_x(u - u_\epsilon)]^2 + \frac{1}{\epsilon} \|\mu\|_{L_t^\infty L_x^2} \|\sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon)\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L_t^2 L_x^\infty} \\ &\leq \alpha \|\mu\|_{L_t^\infty L_x^2}^2 + C(\alpha, \epsilon, \delta, s, v_+, \gamma), \end{aligned}$$

$$\begin{aligned} |I_6| &\leq C \int_0^t \int_{\mathbb{R}} |\mu| \phi_\epsilon \frac{\partial_x v_\epsilon}{v_\epsilon - 1} \left(\frac{\partial_x v_\epsilon |\eta|}{v_\epsilon - 1} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) \frac{\partial_x u_\epsilon}{v_\epsilon - 1} \\ &\leq \frac{C}{\epsilon^2} \left\| \sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon) \right\|_{L_{t,x}^2} \left(\frac{1}{\epsilon} \left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L_{t,x}^2} + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L_{t,x}^2} \right) \\ &\leq C(\epsilon, \delta, s, v_+, \gamma), \end{aligned}$$

$$\begin{aligned} |I_7| &\leq C \int_0^t \int_{\mathbb{R}} |\mu| \phi_\epsilon |\eta| \frac{\partial_x v_\epsilon \partial_x^2(u - u_\epsilon)}{(v_\epsilon - 1)^2} \\ &\leq \frac{C}{\epsilon} \|\eta\|_{L_{t,x}^\infty} \left\| \sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon) \right\|_{L_{t,x}^2} \left(\left\| \sqrt{\phi_\epsilon} \partial_x \mu \right\|_{L_{t,x}^2} + \frac{1}{\epsilon} \left\| \sqrt{\phi_\epsilon} \partial_x(u - u_\epsilon) \right\|_{L_{t,x}^2} \right) \\ &\leq \alpha \left\| \sqrt{\phi_\epsilon} \partial_x \mu \right\|_{L_{t,x}^2}^2 + C(\alpha, \epsilon, \delta, v_+, s, \gamma), \end{aligned}$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension* – 4. *Asymptotic stability of (u_ϵ, v_ϵ)*

and

$$\begin{aligned} |I_8| &\leq C \int_0^t \int_{\mathbb{R}} |\mu| \phi_\epsilon |\eta| \frac{\partial_x v_\epsilon |\partial_x^2 u_\epsilon|}{(v_\epsilon - 1)^2} \leq \frac{C}{\epsilon^3} \left\| \sqrt{\phi_\epsilon} \partial_x (u - u_\epsilon) \right\|_{L_{t,x}^2} \left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L_{t,x}^2} \\ &\leq C(\epsilon, \delta, s, v_+, \gamma). \end{aligned}$$

Hence, coming back to Equation (6.98), by taking $\alpha > 0$ sufficiently small, up to reducing δ , we obtain that there exists $\alpha_0 > 0$ such that

$$\alpha_0 \left(\left\| \mu \right\|_{L_t^\infty L_x^2}^2 + \left\| \sqrt{\phi_\epsilon} \partial_x \mu \right\|_{L_{t,x}^2}^2 \right) \leq \frac{1}{2} \left\| \mu_0 \right\|_{L_x^2}^2 + C(\alpha_0, \epsilon, \delta, s, v_+, \gamma),$$

with $C(\alpha_0, \epsilon, \delta, s, v_+, \gamma) \rightarrow +\infty$ as $\epsilon \rightarrow 0$.

With this estimate, the existence and uniqueness of $u - u_\epsilon$ is classical. It remains to verify that $u - u_\epsilon$ is continuous in time. Let $f \in H^1(\mathbb{R})$ with $\|f\|_{H^1(\mathbb{R})} = 1$. Then using $\partial_t(v - v_\epsilon) = \partial_x(u - u_\epsilon)$, we have

$$\begin{aligned} \langle \partial_t(v - v_\epsilon), f \rangle_{H^{-1} \times H^1} &= (\partial_x(u - u_\epsilon), f) = (u - u_\epsilon, \partial_x f) \\ &\leq \|(u - u_\epsilon)(t)\|_{L_x^2} \end{aligned}$$

and so $\|\partial_t(v - v_\epsilon)\|_{L_t^\infty H_x^{-1}} \leq \|u - u_\epsilon\|_{L_t^\infty L_x^2}$. Since $v - v_\epsilon = \partial_x V$, we have that $\partial_t \partial_x V \in L^\infty(0, T; H^{-1}(\mathbb{R})) \subset L^2(0, T; H^{-1}(\mathbb{R}))$. Since we also have $\partial_x V \in L^2(0, T; H^2(\mathbb{R}))$, it follows (e.g. from Theorem II.5.13. of Franck Boyer et al. 2012) that $\partial_x V \in C([0, T]; H^1(\mathbb{R}))$ and so $V \in C([0, T]; H^2(\mathbb{R}))$. Now since $w = u - \phi_\epsilon(v) \partial_x v$ and $w_\epsilon = u_\epsilon - \phi_\epsilon(v_\epsilon) \partial_x v_\epsilon$, we have that $\partial_x W_0 = u - u_\epsilon + \phi'_\epsilon(v) \partial_x v - \phi'_\epsilon(v_\epsilon) \partial_x v_\epsilon$. After rearranging, we find that

$$u - u_\epsilon = \partial_x W_0 - \phi'_\epsilon(v) \partial_x V + (\phi'_\epsilon(v) - \phi'_\epsilon(v_\epsilon)) \partial_x v_\epsilon.$$

The continuity in time of $W_0, v_\epsilon, \phi'_\epsilon(v)$ and V imply that $u - u_\epsilon \in C([0, T]; H^1(\mathbb{R}))$ as claimed. $\square$

4.2. Bounding $(v - v_\epsilon)$ in $L^1(\mathbb{R})$

We first obtain a L_x^1 bound for $u - u_\epsilon$. Let $\{j_n\}_{n \in \mathbb{N}} \subset C^2(\mathbb{R})$ be a convex approximation of $|\cdot|$ with $|j'_n| \leq C$, $j''_n > 0$. For example, we may take $j_n(z) := \sqrt{z^2 + n^{-1}} - \sqrt{n^{-1}}$. Multiplying (6.89) by $j'_n(u - u_\epsilon)$, we get

$$\begin{aligned} &\int_{\mathbb{R}} j_n(u - u_\epsilon)(t) dx - \int_{\mathbb{R}} j_n(u - u_\epsilon)(0) dx + \int_0^t \int_{\mathbb{R}} j''_n(u - u_\epsilon) \phi_\epsilon(v) |\partial_x(u - u_\epsilon)|^2 \\ &= \int_0^t \int_{\mathbb{R}} j'_n(u - u_\epsilon) ((\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^2 u_\epsilon + (\partial_x \phi_\epsilon(v) - \partial_x \phi_\epsilon(v_\epsilon)) \partial_x u_\epsilon). \end{aligned} \tag{6.99}$$

6. Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet
de suspension – 4. Asymptotic stability of (u_ϵ, v_ϵ)

Using $\partial_x u_\epsilon = -s \partial_x v_\epsilon$, (6.90), (6.91) and (6.58), the right-hand side can be bounded by

$$\begin{aligned} & C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \partial_x V \frac{|\partial_x^2 v_\epsilon|}{v_\epsilon - 1} + C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \left(\frac{\partial_x v_\epsilon |\partial_x V|}{(v_\epsilon - 1)^2} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) \partial_x v_\epsilon \\ & \leq C \|\sqrt{\phi_\epsilon} \partial_x V\|_{L^2_{t,x}} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 v_\epsilon}{v_\epsilon - 1} \right\|_{L^2_{t,x}} \\ & \quad + C \|\sqrt{\phi_\epsilon} \partial_x v_\epsilon\|_{L^2_{t,x}} \left(\left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L^2_{t,x}} + \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L^2_{t,x}} \right). \end{aligned}$$

By Lemma 6.3,

$$\left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 v_\epsilon}{v_\epsilon - 1} \right\|_{L^2_{t,x}} + \|\sqrt{\phi_\epsilon} \partial_x v_\epsilon\|_{L^2_{t,x}} \leq C(s, \gamma, v_+, \epsilon) t^{1/2},$$

Hence we obtain a global bound of the form $C(\epsilon, \delta, s, v_+, \gamma) t^{1/2}$ for the right-hand side. Note that the second term on the left-hand side of (6.99) has a positive sign and therefore can be discarded. Returning to (6.99) and taking $n \rightarrow \infty$, we get

$$\|(u - u_\epsilon)(t)\|_{L^1_x} \leq \|(u - u_\epsilon)(0)\|_{L^1_x} + C(\epsilon, \delta, s, v_+, \gamma) t^{1/2}. \quad (6.100)$$

We now wish to find an estimate for $v - v_\epsilon$. From (6.5) we have that

$$\partial_t(v - v_\epsilon) = \partial_x(u - u_\epsilon).$$

Multiplying this equation by $j'_n(v - v_\epsilon)$ where j_n is defined as above and integrating in space, we have

$$\frac{d}{dt} \int_{\mathbb{R}} j_n(v - v_\epsilon) dx = \int_{\mathbb{R}} j'_n(v - v_\epsilon) \partial_x(u - u_\epsilon) dx \leq \int_{\mathbb{R}} |\partial_x(u - u_\epsilon)| dx. \quad (6.101)$$

Therefore, to obtain a $L^1(\mathbb{R})$ estimate for $v - v_\epsilon$ it is sufficient to bound $\partial_x(u - u_\epsilon)$ in $L^1(\mathbb{R})$. Now, the evolution equation of $\partial_x(u - u_\epsilon)$ can be expressed as

$$\begin{aligned} & \partial_t \partial_x(u - u_\epsilon) - \partial_x(\phi_\epsilon(v) \partial_x^2(u - u_\epsilon)) \\ & = \partial_x((\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^2 u_\epsilon) + \partial_x[\partial_x(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x u] + \partial_x[\partial_x \phi_\epsilon(v_\epsilon) \partial_x(u - u_\epsilon)]. \end{aligned}$$

6. Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet
de suspension – 4. Asymptotic stability of (u_ϵ, v_ϵ)

Multiplying by $j'_n(\partial_x(u - u_\epsilon))$ and integrating by parts where appropriate,

$$\begin{aligned}
& \int_{\mathbb{R}} j_n(\partial_x(u - u_\epsilon))(t) dx - \int_{\mathbb{R}} j_n(\partial_x(u - u_\epsilon))(0) dx \\
& + \int_0^t \int_{\mathbb{R}} j''_n(\partial_x(u - u_\epsilon)) \phi_\epsilon(v) |\partial_x^2(u - u_\epsilon)|^2 \\
& = \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) \partial_x((\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^2 u_\epsilon) \\
& + \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) \partial_x[\partial_x(\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x u] \\
& + \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) \partial_x[\partial_x \phi_\epsilon(v_\epsilon) \partial_x(u - u_\epsilon)] =: \sum_{n=1}^3 I_n.
\end{aligned} \tag{6.102}$$

Firstly using (6.91),

$$\begin{aligned}
|I_1| &= \left| \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) ((\partial_x \phi_\epsilon(v) - \partial_x \phi_\epsilon(v_\epsilon)) \partial_x^2 u_\epsilon + (\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)) \partial_x^3 u_\epsilon) \right| \\
&\leq C \int_0^t \int_{\mathbb{R}} |j'_n(\partial_x(u - u_\epsilon))| \phi_\epsilon(v_\epsilon) \left(|\partial_x^2 v_\epsilon| \left(\frac{\partial_x v_\epsilon |\partial_x V|}{(v_\epsilon - 1)^2} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) + \frac{|\partial_x^3 v_\epsilon|}{v_\epsilon - 1} |\partial_x V| \right) \\
&\leq \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 v_\epsilon}{v_\epsilon - 1} \right\|_{L^2_{t,x}} \left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L^2_{t,x}} \\
&+ \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 v_\epsilon}{v_\epsilon - 1} \right\|_{L^2_{t,x}} \left(\frac{C}{\epsilon} \left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L^2_{t,x}} + \left\| \sqrt{\phi_\epsilon} \partial_x^2 V \right\|_{L^2_{t,x}} \right) \\
&\leq C(\epsilon, \delta, s, v_+, \gamma) t^{1/2}.
\end{aligned}$$

Then

$$\begin{aligned}
I_2 &= \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) \partial_x^2[\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)] \partial_x u + \int_0^t \int_{\mathbb{R}} j'_n(\partial_x(u - u_\epsilon)) \partial_x[\phi_\epsilon(v) - \phi_\epsilon(v_\epsilon)] \partial_x^2 u \\
&=: K_1 + K_2.
\end{aligned}$$

Using (6.92),

$$\begin{aligned}
|K_1| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \left(\frac{(\partial_x v_\epsilon)^2 |\partial_x V|}{(v_\epsilon - 1)^3} + \frac{\partial_x v_\epsilon |\partial_x^2 V|}{(v_\epsilon - 1)^2} + \left| \frac{\partial_x^2 V}{v_\epsilon - 1} \right|^2 + \frac{|\partial_x^3 V|}{v_\epsilon - 1} \right) |\partial_x u| \\
&\leq C \left\| \sqrt{\phi_\epsilon} \partial_x u \right\|_{L^2_{t,x}} \left(\left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L^2_{t,x}} + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L^2_{t,x}} + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^3 V}{v_\epsilon - 1} \right\|_{L^2_{t,x}} \right. \\
&\quad \left. + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L^2 L^\infty} \left\| \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L^\infty L^2} \right),
\end{aligned}$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension* – 4. *Asymptotic stability of (u_ϵ, v_ϵ)*

and the triangle inequality

$$\|\sqrt{\phi_\epsilon} \partial_x u\|_{L^2_{t,x}} \leq \|\sqrt{\phi_\epsilon} \partial_x (u - u_\epsilon)\|_{L^2_{t,x}} + \|\sqrt{\phi_\epsilon} \partial_x u_\epsilon\|_{L^2_{t,x}}$$

enables to bound K_1 . For K_2 , (6.91) yields

$$\begin{aligned} |K_2| &\leq C \int_0^t \int_{\mathbb{R}} \phi_\epsilon(v_\epsilon) \left(\frac{\partial_x v_\epsilon |\partial_x V|}{(v_\epsilon - 1)^2} + \frac{|\partial_x^2 V|}{v_\epsilon - 1} \right) |\partial_x^2 u| \\ &\leq \left(\left\| \sqrt{\phi_\epsilon} \partial_x V \right\|_{L^2_{t,x}} + \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 V}{v_\epsilon - 1} \right\|_{L^2_{t,x}} \right) \left(\left\| \sqrt{\phi_\epsilon} \partial_x^2 (u - u_\epsilon) \right\|_{L^2_{t,x}} + \left\| \sqrt{\phi_\epsilon} \partial_x^2 u_\epsilon \right\|_{L^2_{t,x}} \right). \end{aligned}$$

For I_3 , we decompose

$$\begin{aligned} |I_3| &= C \left| \int_0^t \int_{\mathbb{R}} j'_n \left[(\phi''_\epsilon(v_\epsilon) (\partial_x v_\epsilon)^2 + \phi'_\epsilon(v_\epsilon) \partial_x^2 v_\epsilon) \partial_x (u - u_\epsilon) + \phi'_\epsilon(v_\epsilon) \partial_x v_\epsilon \partial_x^2 (u - u_\epsilon) \right] \right| \\ &\leq C \left[\left\| \frac{\phi''_\epsilon(v_\epsilon) v'_\epsilon}{\phi_\epsilon(v_\epsilon)} \right\|_{L^\infty_{t,x}} \left\| \sqrt{\phi_\epsilon} \partial_x v_\epsilon \right\|_{L^2_{t,x}} + \left\| \frac{\phi'_\epsilon(v_\epsilon) (v_\epsilon - 1)}{\phi_\epsilon(v_\epsilon)} \right\|_{L^\infty_{t,x}} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 v_\epsilon}{v_\epsilon - 1} \right\|_{L^2_{t,x}} \right] \times \\ &\quad \sqrt{\phi_\epsilon} \partial_x (u - u_\epsilon) \|_{L^2_{t,x}} \\ &\quad + C \left\| \frac{\phi'_\epsilon(v_\epsilon) (v_\epsilon - 1)}{\phi_\epsilon(v_\epsilon)} \right\|_{L^\infty_{t,x}} \left\| \sqrt{\phi_\epsilon} \partial_x v_\epsilon \right\|_{L^2_{t,x}} \left\| \sqrt{\phi_\epsilon} \frac{\partial_x^2 (u - u_\epsilon)}{v_\epsilon - 1} \right\|_{L^2_{t,x}}. \end{aligned}$$

Passing to the limit $n \rightarrow \infty$ with the help of Fatou's lemma gives us

$$\|\partial_x (u - u_\epsilon)(t)\|_{L^1_x} \leq \|\partial_x (u - u_\epsilon)(0)\|_{L^1_x} + C(\epsilon, \delta, s, v_+, \gamma) t^{1/2}. \quad (6.103)$$

We have now shown that $(u - u_\epsilon)(t)$, $(v - v_\epsilon)(t)$ belong to $L^1(\mathbb{R})$ for any $t > 0$. Note that since the equations for both quantities are conservative, we actually have that $(u - u_\epsilon)(t)$, $(v - v_\epsilon)(t) \in L^1_0(\mathbb{R})$. This marks the end of the proof of Proposition 6.3.

4.3. Concluding the proofs of Theorem 6.2 and Corollary 6.1

We are now in a position to justify the equivalence between the original and integrated systems, which will allow us to conclude the proof of Theorem 6.2. Firstly, consider the original system with some initial data (u_0, v_0) which satisfies the assumptions of Theorem 6.2, and let (V, W) be the corresponding solution of the integrated system (6.25). Since we defined $(u, v) := (u_\epsilon + \partial_x U, v_\epsilon + \partial_x V)$, it follows that (u, v) is a solution to the original system. Moreover, Lemma 6.8 tells us that $\partial_x U \in C([0, T]; H^1(\mathbb{R}))$ and Theorem 6.1 tells us that $\partial_x V \in C([0, T]; H^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R}))$. Thus, $(u, v) \in (u_\epsilon, v_\epsilon) + (C([0, T]; H^1(\mathbb{R})))^2$ solves the original system and using Proposition 6.3 we have that $(u, v)(t) \in L^1_0(\mathbb{R})$ for any $t \in (0, T]$. It remains to show that the solution is unique. To this end, suppose that (u, v) is another solution to the original system with initial data (u_0, v_0) satisfying the assumptions of Theorem 6.2. Then we may write $(u, v) = (u_\epsilon, v_\epsilon) + (f, g)$ for some $f, g \in C([0, T]; H^1(\mathbb{R}) \cap L^1_0(\mathbb{R}))$. Now, define the

integrated quantities

$$U(t, x) := \int_{-\infty}^x (u(t, z) - u_\epsilon(t, z)) dz, \quad V(t, x) := \int_{-\infty}^x (v(t, z) - v_\epsilon(t, z)) dz,$$

and

$$W := U + \phi_\epsilon(v) - \phi_\epsilon(v_\epsilon).$$

Then (V, W) is a solution of the integrated system (6.25). Furthermore, thanks to the hypotheses on the initial data (u, v) we have that $\partial_x V \in C([0, T]; H^1(\mathbb{R}) \cap L_0^1(\mathbb{R})) \cap L^2(0, T; H^2(\mathbb{R}))$. Therefore $V \in \mathcal{X}$. Moreover, due to the assumptions on the initial data, we in fact have $V \in B_\delta$ and so since $\mathcal{A}^\epsilon$ is a contraction on B_δ we easily deduce that V is uniquely determined by the initial data. Since W is constant in time, we conclude that the pair (W, V) is unique and therefore so is (u, v) .

Finally, let us explain how Corollary 6.1 is obtained. With the additional assumption (6.18), the inequality (6.73) becomes

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\sum_{k=0}^2 c_k \epsilon^{2k} \left[E_k(t; V_2) + \int_0^t D_k(\tau; V_2) d\tau \right] \right) \\ & \leq C(E_0(0; V_2) + \epsilon^2 E_1(0; V_2) + \epsilon^4 E_2(0; V_2)) + \frac{C}{\epsilon^{3/2}} \|V_1\|_{\mathcal{X}}^2 \|V_2\|_{\mathcal{X}}, \end{aligned} \quad (6.104)$$

and so we may take $T = +\infty$ which gives us a solution defined on $\mathbb{R}_+ \times \mathbb{R}$. Then using (6.48) we have that there exists $C > 0$ independent of time such that

$$\|(v - v_\epsilon)(t)\|_{L_x^\infty} \leq C \|(v - v_\epsilon)(t)\|_{L_x^2}^{1/2} \|\partial_x(v - v_\epsilon)(t)\|_{L_x^2}^{1/2}, \quad (6.105)$$

where the right hand-side tends to 0 as $t \rightarrow \infty$ since $V \in L^2(\mathbb{R}_+; H^2(\mathbb{R}))$. This shows that $(v - v_\epsilon)(t) \rightarrow 0$ as $t \rightarrow \infty$. Similarly, the bounds obtained in Lemma 6.8 imply that

$$\|(u - u_\epsilon)(t)\|_{H_x^1} \rightarrow 0 \text{ as } t \rightarrow \infty$$

and so we also find that $(u - u_\epsilon)(t) \rightarrow 0$ as $t \rightarrow \infty$, which completes the proof.

5. Appendix

5.1. Proof that the map $\mathcal{A}^\epsilon$ is well defined

We give here a proof that the map $\mathcal{A}^\epsilon$ of Section 3.3 is well defined, i.e. we show that Equation (6.43):

$$\begin{cases} \partial_t V_2 - \partial_x W_0 - \partial_x(\phi_\epsilon(v_\epsilon) \partial_x V_2) = \partial_x H(\partial_x V_1), \\ V_2|_{t=0} = V_0, \end{cases}$$

is well posed. Equation (6.43) rewrites as

$$\begin{cases} \partial_t V_2 = \partial_x(\phi_\epsilon \partial_x V_2) + f, \\ V_2|_{t=0} = V_0, \end{cases} \quad (6.106)$$

where $f = \partial_x H(\partial_x V_1) + \partial_x W_0$. Since the operator $\partial_x(\phi_\epsilon \partial_x \cdot)$ is uniformly elliptic, one can hope that it is the infinitesimal generator of an analytic evolution operator $(U(t, \tau))_{t, \tau \geq 0}$. The unique solution of (6.106) is then given by

$$V_2(t) = U(t, 0) V_0 + \int_0^t U(t, \tau) f(\tau) d\tau.$$

The proof that $\partial_x(\phi_\epsilon \partial_x \cdot)$ generates an analytic evolution operator is classical. Note that one has to be careful because the coefficient $\phi_\epsilon = \phi_\epsilon(t, x)$ depends on time and is unbounded. In order to simplify the situation, we take advantage of the fact that ϕ_ϵ is a function of the variable $\xi := x - st$. A change of variable $(t, x) \mapsto (t, \xi) \in (0, T) \times \mathbb{R}$ then yields that (6.106) can be written as

$$\begin{cases} \partial_t V_2 = \partial_\xi(\phi_\epsilon(\xi) \partial_\xi V_2) + s \partial_\xi V_2 + f, \\ V_2|_{t=0} = V_0, \end{cases}$$

Hence in these coordinates, the operator $\partial_\xi(\phi_\epsilon(\xi) \partial_\xi \cdot) + s \partial_\xi$ is independant of time. Consider the Hilbert space $H = L^2(\mathbb{R}, \mathbb{C})$, and define the unbounded operator

$$T_\epsilon V := \phi_\epsilon^{1/2} \partial_\xi V, \quad \text{with} \quad D(T_\epsilon) = C_c^\infty(\mathbb{R}).$$

Let T be the closure of T_ϵ , i.e.

$$D(T) = \{V \in H, \phi_\epsilon^{1/2} \partial_\xi V \in H, \exists V_n \in C_c^\infty(\mathbb{R}), V_n \rightarrow V, \phi_\epsilon^{1/2} \partial_\xi V_n \rightarrow \phi_\epsilon^{1/2} \partial_\xi V\}.$$

The adjoint T^* of T is given by

$$D(T^*) = \{V \in H, \partial_\xi(\phi_\epsilon^{1/2} V) \in H\}, \quad \text{with} \quad T^* V = -\partial_\xi(\phi_\epsilon^{1/2} V).$$

Indeed, if $V \in D(T^*)$, then $T^* V = -\partial_\xi(\phi_\epsilon^{1/2} V)$ in the sense of distributions. Conversely, if V and $\partial_\xi(\phi_\epsilon^{1/2} V) \in H$, then $\langle -\partial_\xi(\phi_\epsilon^{1/2} V), \psi \rangle_H = \langle V, T\psi \rangle_H$ for every $\psi \in C_c^\infty(\mathbb{R})$, and the same is true for every $\psi \in D(T)$ by density.

Since T is closed and densely defined, $T^* T$ defined by

$$D(T^* T) = \{V \in D(T), TV \in D(T^*)\}, \quad T^* TV = T^*(TV) = -\partial_\xi(\phi_\epsilon(\xi) \partial_x V)$$

is self-adjoint and positive (see Theorem 13.13 in Rudin 1991, Chapter 13). It follows that $-T^* T$ is the infinitesimal generator of an analytic semigroup. Now define the closed operator $B := s \partial_\xi$, with $D(B) = H^1(\mathbb{R})$. Since there exists $\alpha > 0$ such that $\phi_\epsilon \geq \alpha$, we deduce that $D(T^* T) \subset D(T) \subset D(B)$. Furthermore, for every $\epsilon > 0$, for every $V \in$

$D(T^*T)$, one has that

$$\begin{aligned}\|BV\| &\leq \frac{s}{\alpha} \|TV\| = \frac{s}{\alpha} |\langle V, T^*TV \rangle_H|^{1/2} \leq \frac{s}{\alpha} \|V\|^{1/2} \|T^*TV\|^{1/2} \\ &\leq \epsilon \|T^*TV\| + \frac{s^2}{4\alpha^2\epsilon} \|V\|,\end{aligned}$$

by Young's inequality. By Theorem 12.37 in Renardy et al. 2006, it follows that $-T^*T+B$ is the infinitesimal generator of an analytic semigroup.

Note that in fact it is possible to prove that

$$D(T) = \{V \in H, \phi_\epsilon^{1/2} \partial_x V \in H\}.$$

Indeed, let θ a smooth function such that $0 \leq \theta \leq 1$, $\theta|_{[-\frac{1}{3}, \frac{1}{3}]} \equiv 1$, $\text{supp}(\theta) \subset [-1, 1]$ and $\int_{\mathbb{R}} \theta = 1$. For $V \in H$ with $\phi_\epsilon^{1/2} \partial_\xi V \in H$, define

$$V_n(\xi) := \theta\left(\frac{\xi}{n}\right) (\rho_n * V)(\xi),$$

where $\rho_n := n\theta(n\cdot) \in C_c^\infty(\mathbb{R})$ is an approximation of unity. Then $V_n \in C_c^\infty(\mathbb{R})$ and $V_n \rightarrow V$ in L^2 . Furthermore,

$$\phi_\epsilon^{1/2} \partial_\xi V_n = \frac{\phi_\epsilon^{1/2}}{n} \theta'\left(\frac{\xi}{n}\right) (\rho_n * V)(\xi) + \phi_\epsilon^{1/2} \theta\left(\frac{\xi}{n}\right) (\rho_n * \partial_\xi V)(\xi).$$

Since $\phi_\epsilon^{1/2}(\xi) \leq C(1 + |\xi|^{1/2+1/(2\gamma)})$ and θ is compactly supported,

$$\left| \frac{\phi_\epsilon^{1/2}}{n} \theta'\left(\frac{\xi}{n}\right) \right| \leq C n^{1/(2\gamma)-1/2} \mathbf{1}_{n/3 \leq |\xi| \leq n}.$$

Hence

$$\frac{\phi_\epsilon^{1/2}}{n} \theta'\left(\frac{\xi}{n}\right) (\rho_n * V)(\xi) \rightarrow 0 \quad \text{and} \quad \phi_\epsilon^{1/2} \theta\left(\frac{\xi}{n}\right) (\rho_n * \partial_\xi V)(\xi) \rightarrow \phi_\epsilon^{1/2} \partial_\xi V.$$

Hence $V \in D(T)$.

5.2. Proof of Lemma 6.5

We give here the proof of Lemma 6.5.

Proof. We first do the computations of the estimate for $\mathcal{L}_\epsilon$. We compute that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\eta) \eta &= \int_0^t \int_{\mathbb{R}} \frac{s\eta^2 v'_\epsilon}{v_\epsilon - 1} dx d\tau + \int_0^t \int_{\mathbb{R}} \frac{\phi_\epsilon(v_\epsilon) v'_\epsilon}{v_\epsilon - 1} \eta \left(\partial_x \eta + \frac{\eta v'_\epsilon}{v_\epsilon - 1} \right) dx d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}} \eta \partial_x \left(\frac{\eta \phi_\epsilon(v_\epsilon) v'_\epsilon}{v_\epsilon - 1} \right) dx d\tau + \int_0^t \int_{\mathbb{R}} \frac{\eta}{v_\epsilon - 1} \partial_x (\phi'_\epsilon(v_\epsilon) v'_\epsilon (v_\epsilon - 1) \eta) dx d\tau \\ &=: \sum_{k=1}^4 I_k. \end{aligned}$$

Let us fix $\alpha \in (0, 1)$. For I_1 , we exploit the smallness of v'_ϵ (i.e. (6.47)) to get

$$|I_1| = \left| \int_0^t \int_{\mathbb{R}} \frac{s\eta^2 v'_\epsilon}{v_\epsilon - 1} dx d\tau \right| \leq \left\| \frac{s v'_\epsilon}{\phi_\epsilon(v_\epsilon - 1)^3} \right\|_{L_{t,x}^\infty} \int_0^t \int_{\mathbb{R}} |\sqrt{\phi_\epsilon} \partial_x V|^2 \leq \frac{C}{\epsilon^2} \int_0^t D_0(\tau; V) d\tau.$$

Next,

$$\begin{aligned} |I_2| &= \left| \int_0^t \int_{\mathbb{R}} \frac{v'_\epsilon \phi_\epsilon}{v_\epsilon - 1} \eta \left(\partial_x \eta + \frac{\eta v'_\epsilon}{v_\epsilon - 1} \right) dx d\tau \right| \\ &\leq \int_0^t \int_{\mathbb{R}} \left| \frac{v'_\epsilon \phi_\epsilon}{v_\epsilon - 1} \right| |\eta| |\partial_x \eta| + \int_0^t \int_{\mathbb{R}} \left| \frac{v'_\epsilon \phi_\epsilon}{v_\epsilon - 1} \frac{v'_\epsilon \partial_x V}{(v_\epsilon - 1)^2} \frac{\partial_x V}{v_\epsilon - 1} \right| \\ &\leq \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L_{t,x}^2} + \left\| \frac{(v'_\epsilon)^2}{(v_\epsilon - 1)^4} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L_{t,x}^2}^2 \\ &\leq \frac{C}{\epsilon} \left(\int_0^t D_0(\tau; V) d\tau \right)^{\frac{1}{2}} \left(\int_0^t D_1(\tau; V) d\tau \right)^{\frac{1}{2}} + \frac{C}{\epsilon^2} \int_0^t D_0(\tau; V) d\tau \\ &\leq \frac{\alpha}{3} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_0(\tau; V) d\tau, \end{aligned}$$

where we used Young's inequality for the last line. Similarly,

$$\begin{aligned} |I_3| &= \left| \int_0^t \int_{\mathbb{R}} \eta \partial_x \left(\frac{\eta \phi_\epsilon(v_\epsilon) v'_\epsilon}{v_\epsilon - 1} \right) dx d\tau \right| \\ &\leq \int_0^t \int_{\mathbb{R}} |\partial_x \eta| \left| \frac{\eta v'_\epsilon \phi_\epsilon}{v_\epsilon - 1} \right| \leq \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L_{t,x}^2} \\ &\leq \frac{\alpha}{3} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_0(\tau; V) d\tau. \end{aligned}$$

To deal with the last term, we first integrate by parts to get

$$\begin{aligned} I_4 &= \int_0^t \int_{\mathbb{R}} \frac{\eta}{v_\epsilon - 1} \partial_x (\phi'_\epsilon(v_\epsilon) v'_\epsilon (v_\epsilon - 1) \eta) dx d\tau \\ &= - \int_0^t \int_{\mathbb{R}} \frac{\partial_x \eta}{v_\epsilon - 1} v'_\epsilon \phi'_\epsilon \partial_x V + \int_0^t \int_{\mathbb{R}} \frac{(v'_\epsilon)^2 \eta}{(v_\epsilon - 1)^2} \phi'_\epsilon \partial_x V. \end{aligned}$$

Then in the same spirit as the previous estimates and with the estimate (6.46) on ϕ'_ϵ ,

$$\begin{aligned} |I_4| &\leq \left\| \frac{v'_\epsilon}{(v_\epsilon - 1)^2} \right\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L^2_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L^2_{t,x}} + \left\| \frac{(v'_\epsilon)^2}{(v_\epsilon - 1)^4} \right\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L^2_{t,x}}^2 \\ &\leq \frac{\alpha}{3} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_0(\tau; V) d\tau. \end{aligned}$$

We finally put all terms together to obtain that

$$\left| \int_0^t \int_{\mathbb{R}} \mathcal{L}_\epsilon(\eta) \eta \right| \leq \sum_{k=1}^4 |I_k| \leq \alpha \int_0^t D_1(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_0(\tau; V) d\tau.$$

We now move on to the proof of the second estimate. Again, we fix $\alpha \in (0, 1)$ and compute that

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} \partial_x \eta \mathcal{C}_\epsilon(\eta) &= - \int_0^t \int_{\mathbb{R}} \partial_x \phi_\epsilon \partial_x^2 \eta \partial_x \eta + \int_0^t \int_{\mathbb{R}} \eta \partial_x \eta \partial_x \left(\frac{s v'_\epsilon}{v_\epsilon - 1} \right) \\ &\quad + \int_0^t \int_{\mathbb{R}} (\partial_x \eta)^2 \partial_x \left(\phi_\epsilon \frac{v'_\epsilon}{v_\epsilon - 1} \right) + \int_0^t \int_{\mathbb{R}} \eta \partial_x \eta \partial_x \left(\frac{(v'_\epsilon)^2 \phi_\epsilon(v_\epsilon)}{(v_\epsilon - 1)^2} \right) \\ &\quad + \int_0^t \int_{\mathbb{R}} \partial_x \eta \partial_x \left(\eta \partial_x \left(\frac{v'_\epsilon \phi_\epsilon(v_\epsilon)}{v_\epsilon - 1} \right) \right) - \int_0^t \int_{\mathbb{R}} \partial_x \eta \frac{v'_\epsilon \partial_x (\phi'_\epsilon(v_\epsilon) v'_\epsilon (v_\epsilon - 1) \eta)}{(v_\epsilon - 1)^2} \\ &\quad + \int_0^t \int_{\mathbb{R}} \frac{\partial_x \eta}{v_\epsilon - 1} \partial_x (\partial_x (v'_\epsilon \phi'_\epsilon(v_\epsilon) (v_\epsilon - 1)) \eta) =: \sum_{n=1}^7 K_n. \end{aligned}$$

We can estimate each of these terms using the same strategy as our estimates for $I_1 - I_4$.

This leads to

$$\begin{aligned}
 |K_1| &\leq \left\| \frac{\partial_x \phi_\epsilon}{\phi_\epsilon} \right\|_{L_{t,x}^\infty} \|\sqrt{\phi_\epsilon} \partial_x^2 \eta\|_{L_{t,x}^2} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L_{t,x}^2} \leq \frac{\alpha}{6} \int_0^t D_2(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_1(\tau; V) d\tau, \\
 |K_2| &= \left| \int_0^t \int_{\mathbb{R}} \frac{s v'_\epsilon}{v_\epsilon - 1} ((\partial_x \eta)^2 + \eta \partial_x^2 \eta) \right| \\
 &\leq \frac{\alpha}{6} \int_0^t D_2(\tau; V) d\tau + \frac{C}{\epsilon^2} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\alpha \epsilon^4} \int_0^t D_0(\tau; V) d\tau, \\
 |K_3| &= 2 \left| \int_0^t \int_{\mathbb{R}} \phi_\epsilon \frac{v'_\epsilon}{v_\epsilon - 1} \partial_x^2 \eta \partial_x \eta \right| \leq \frac{\alpha}{6} \int_0^t D_2(\tau; V) d\tau + \frac{C}{\alpha \epsilon^2} \int_0^t D_1(\tau; V) d\tau, \\
 |K_4| &= \left| \int_0^t \int_{\mathbb{R}} \frac{(v'_\epsilon)^2 \phi_\epsilon}{(v_\epsilon - 1)^2} (\eta \partial_x^2 \eta + (\partial_x \eta)^2) \right| \\
 &\leq \frac{\alpha}{6} \int_0^t D_2(\tau; V) d\tau + \frac{C}{\epsilon^2} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\epsilon^4} \int_0^t D_0(\tau; V) d\tau, \\
 |K_5| &= \left| \int_0^t \int_{\mathbb{R}} \partial_x \left(\phi_\epsilon \frac{v'_\epsilon}{v_\epsilon - 1} \right) \eta \partial_x^2 \eta \right|,
 \end{aligned}$$

and we compute with the estimate (6.46) and Lemma 6.1 that

$$\left| \partial_x \left(\phi_\epsilon \frac{v'_\epsilon}{v_\epsilon - 1} \right) \right| \leq \frac{C}{\epsilon^2} \phi_\epsilon (v_\epsilon - 1)^{2\gamma} \leq \frac{C}{\epsilon^2} \phi_\epsilon,$$

hence

$$|K_5| \leq \frac{\alpha}{6} \int_0^t D_2(\tau; V) d\tau + \frac{C}{\alpha \epsilon^4} \int_0^t D_0(\tau; V) d\tau.$$

Next,

$$K_6 = - \int_0^t \int_{\mathbb{R}} (\partial_x \eta)^2 \frac{v'_\epsilon \phi'_\epsilon(v_\epsilon) v'_\epsilon (v_\epsilon - 1)}{(v_\epsilon - 1)^2} - \int_0^t \int_{\mathbb{R}} \eta \partial_x \eta \frac{v'_\epsilon \partial_x (\phi'_\epsilon(v_\epsilon) v'_\epsilon (v_\epsilon - 1))}{(v_\epsilon - 1)^2}.$$

With the estimate (6.46) and Lemma 6.1, we see that

$$|\partial_x [\phi'_\epsilon v'_\epsilon (v_\epsilon - 1)]| \leq \frac{C}{\epsilon} (v_\epsilon - 1)^\gamma. \quad (6.107)$$

Hence

$$|K_6| \leq \frac{C}{\epsilon^2} \int_0^t D_1(\tau; V) d\tau + \frac{C}{\epsilon^4} \int_0^t D_0(\tau; V) d\tau.$$

To estimate K_7 , we first integrate by parts to obtain

$$K_7 = - \int_0^t \int_{\mathbb{R}} \frac{\eta \partial_x^2 \eta}{v_\epsilon - 1} \partial_x [\phi'_\epsilon v'_\epsilon (v_\epsilon - 1)] + \int_0^t \int_{\mathbb{R}} \frac{\eta_2 \partial_x \eta}{(v_\epsilon - 1)^2} v'_\epsilon \partial_x [\phi'_\epsilon v'_\epsilon (v_\epsilon - 1)].$$

6. *Stabilité d'ondes progressives partiellement congestionnées pour un modèle jouet de suspension* – 5. Appendix

Then using again (6.107) and the Holder and Young inequalities,

$$\begin{aligned}
 |K_7| &\leq \frac{C}{\epsilon} \|\sqrt{\phi_\epsilon} \partial_x V\|_{L^2_{t,x}} \left(\left\| \frac{(v_\epsilon - 1)^{\gamma-2}}{\phi_\epsilon} \right\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x^2 \eta\|_{L^2_{t,x}} + \left\| \frac{v'_\epsilon (v_\epsilon - 1)^{\gamma-3}}{\phi_\epsilon} \right\|_{L^\infty_{t,x}} \|\sqrt{\phi_\epsilon} \partial_x \eta\|_{L^2_{t,x}} \right) \\
 &\leq \frac{C}{\epsilon^2} \int_0^t D_1(\tau; V) \, d\tau + \frac{C}{\alpha \epsilon^4} \int_0^t D_0(\tau; V) \, d\tau + \frac{\alpha}{6} \int_0^t D_2(\tau; V) \, d\tau.
 \end{aligned}$$

Summing $|K_1|, \dots, |K_7|$ finally yields the desired estimate. $\square$

Conclusion

Dans cette thèse, plusieurs contributions ont été apportées concernant la dérivation et l'analyse mathématique de modèles de fluides complexes, motivées principalement par la description des phénomènes d'avalanche.

Dans la première partie, on a présenté des résultats concernant la structure mathématique de modèles moyennés de fluides parfaits pour des écoulements cisailés ou turbulents. L'hétérogénéité de la vitesse est prise en compte grâce au tenseur de Reynolds, défini à partir des secondes corrélations des fluctuations de vitesse.

Un premier résultat, qui concerne les écoulements monophasés et barotropes, est une caractérisation mathématique de la symétrisabilité (au sens de Friedrichs) des équations moyennées de Reynolds. On montre en effet que ce système d'équations est symétrisable en dimension 2 ou 3 si, et seulement si, le tenseur de Reynolds s'écrit comme une matrice scalaire, ou le gradient de la pression s'annule. Ce résultat permet donc de compléter la compréhension mathématique de ces équations. En effet, ce système étant non conservatif, il ne peut pas être symétrisé simplement à l'aide d'une entropie mathématique, et sa symétrisabilité était jusqu'alors inconnue.

Un second résultat concerne les écoulements diphasiques, c'est-à-dire formés par le mélange de deux phases. Pour ces écoulements, une manière de dériver un modèle en accord avec les lois de la physique est de définir un lagrangien à partir de l'énergie du mélange, et d'appliquer le principe de l'action stationnaire de Hamilton. Dans le cas d'un modèle à deux vitesses pour lequel l'énergie du mélange est simplement définie comme la somme des énergies de chaque phase, le système obtenu par principe de Hamilton n'est pas hyperbolique, et des vitesses caractéristiques complexes apparaissent lorsque les deux vitesses sont proches. Une contribution de cette thèse est l'obtention d'une série de modèles d'écoulements diphasiques par principe de Hamilton, qui sont hyperboliques lorsque les deux vitesses sont proches. L'hyperbolicité des modèles est garantie par la prise en compte du tenseur de Reynolds dans l'énergie du mélange. Une condition de compatibilité naturelle permet également de caractériser les énergies de mélange admissibles, pour lesquelles le lagrangien associé permet d'obtenir des équations hyperboliques lorsque les deux vitesses sont proches. Cette contribution ouvre la voie à des travaux supplémentaires qui pourraient compléter ce résultat. D'un point de vue mathématique, l'étude de l'hyperbolicité des équations pourrait être étendue au cas général de vitesses prenant des valeurs arbitraires. Il serait en effet pertinent de caractériser avec précision le domaine d'hyperbolicité des équations. D'un point de vue de modélisation, plusieurs perspectives existent également. Il serait tout d'abord intéressant de dériver des équations similaires à celles qui ont été proposées dans le chapitre 2 à partir de modèles de mélange microscopiques

ou de modèles de phases séparées avec interfaces. Afin de valider la pertinence des modèles obtenus pendant cette thèse, un prolongement intéressant de ces travaux serait de comparer les prédictions théoriques correspondantes avec des résultats expérimentaux dans un cas d'application concrète.

Dans la deuxième partie de la thèse, on a présenté des modèles décrivant un écoulement granulaire en couche mince sur un plan incliné. Ces modèles sont obtenus à partir de la moyenne sur la profondeur d'équations constitutives données par une rhéologie frictionnelle empirique (la rhéologie $\mu(I)$). Ils sont constitués de trois équations, donnant l'évolution de trois variables caractérisant l'écoulement : la hauteur, la vitesse moyenne, et l'ensrophie, qui est la version unidimensionnelle du tenseur de Reynolds et qui permet ici de modéliser les effets de cisaillement vertical au sein de l'écoulement. Ces modèles ont une structure hyperbolique avec des termes sources, et satisfont un bilan d'énergie, ce qui garantit leur caractère bien posé et une résolution numérique à l'aide de schémas robustes.

Un premier modèle a été présenté au chapitre 3. Il a été écrit à partir de la rhéologie $\mu(I)$ incompressible. Sa dérivation a été effectuée à l'aide d'un développement asymptotique, ce qui a permis de calculer avec précision l'écart entre la loi de friction granulaire classique, qui est utilisée pour les écoulements moyennés sur la profondeur, et l'expression de la friction au fond donnée par la rhéologie $\mu(I)$. La prise en compte de cet écart a permis d'écrire pour la première fois un modèle consistant jusqu'à l'ordre 1 avec cette rhéologie. La connaissance du cisaillement via l'ensrophie permet également d'obtenir une reconstruction du profil de vitesse sur la profondeur. Une version d'ordre deux du modèle, comprenant des termes de diffusion, a également été proposée. L'étude de la stabilité linéaire du modèle et de la version d'ordre deux a été effectuée et appliquée à l'instabilité des ondes de surface. On a tout d'abord montré que cette instabilité apparaît lorsque le nombre de Froude dépasse un seuil critique. Pour les deux versions du modèle, ce seuil est le même que celui prédit théoriquement par la rhéologie $\mu(I)$, grâce à la consistance des équations jusqu'à l'ordre 1. Une étude plus poussée de la relation de dispersion a également montré que le modèle avec diffusion linéarisé prédit correctement le taux de croissance et la vitesse de phase, ainsi que la fréquence de coupure à partir de laquelle l'instabilité disparaît. Des ondes de surface ont été calculées à partir de la résolution numérique des deux modèles. Les limites de ces modèles ont également été étudiées. Un défaut majeur de ces modèles est qu'ils ont été écrits à partir des propriétés des écoulements proches d'une solution stationnaire et uniforme de la rhéologie. Pour un écoulement fortement instationnaire et/ou non uniforme, rien ne garantit la validité du développement asymptotique, et donc du modèle proposé. De plus, les écoulements stationnaires et uniformes n'existent théoriquement que pour une pente comprise entre les limites μ_1 et μ_2 , qui sont deux paramètres de la rhéologie $\mu(I)$. Ces paramètres correspondent à des pentes à partir desquelles une transition de l'état du milieu granulaire peut être observée. En effet, pour une pente inférieure à μ_1 , l'écoulement s'arrête (transition liquide/solide). De même, pour une pente supérieure à μ_2 , l'écoulement accélère, la fraction volumique décroît et la dynamique est dominée par des collisions binaires

(transition liquide/gaz). Pour une pente qui n'est pas comprise entre ces deux valeurs, certains coefficients du modèle ne sont pas définis, et leur extrapolation tels quels pourrait donner des comportements incohérents physiquement et mathématiquement. Il est donc important de pouvoir écrire une version régularisée du modèle, qui puisse être utilisée même lorsque la pente n'est pas comprise entre μ_1 et μ_2 .

Une version régularisée du modèle sans diffusion a donc été présentée au chapitre 4. Un coefficient de régularisation noté f a été introduit. Ce coefficient vaut 1 pour un écoulement stationnaire et uniforme sur une pente comprise entre μ_1 et μ_2 , et 0 lorsque l'écoulement est loin de cet équilibre (pente en dehors de l'intervalle $]\mu_1, \mu_2[$, front sec, écoulement à l'arrêt). Ce coefficient permet donc de quantifier la distance entre la solution et un écoulement stationnaire et uniforme. Il permet de faire la transition entre le modèle consistant présenté au chapitre 3 lorsque $f = 1$, et un modèle réduit ne tenant compte que des effets physiques essentiels (poids, friction, pression hydrostatique, enstrophie) lorsque $f = 0$. Notons que le modèle régularisé est écrit de telle sorte à ce que la consistance à l'ordre 1 avec la rhéologie $\mu(I)$ soit préservée. En particulier, le modèle linéarisé au voisinage d'un écoulement stationnaire et uniforme reste inchangé, ce qui permet de retrouver la relation de dispersion du modèle présenté au chapitre 3 et les bonnes propriétés qui en découlent. De plus, des comparaisons de simulations numériques avec des données expérimentales montrent que le modèle régularisé permet de prédire quantitativement des fronts secs, des arrêts d'avalanche et des écoulements accélérés. Un prolongement de ce travail serait d'écrire également une version régularisée pour le modèle avec diffusion présenté au chapitre 3.

Un troisième modèle moyenné sur la profondeur a été proposé au chapitre 5. Ce modèle a été dérivé selon la même méthode que les modèles précédents, mais en considérant au départ une version de la rhéologie $\mu(I)$ prenant en compte des effets de dilatance, i.e. pour laquelle la fraction volumique solide peut varier et décroître avec le nombre inertiel. Comme ce modèle a été écrit pour un écoulement compressible, les opérateurs de moyennisation sur la profondeur ont été remplacés par des moyennes de Favre, qui prennent en compte les variations de la densité selon la profondeur. En particulier, le modèle ne calcule pas la hauteur, mais une autre variable appelée le "hold-up" qui correspond à l'intégrale sur la profondeur de la fraction de volume solide, normalisée de manière à être égale à la hauteur lorsque l'écoulement est homogène. Ce modèle a exactement la même structure que le modèle présenté au chapitre 3, et peut être écrit de la même façon en redéfinissant une constante de manière à prendre en compte la dilatance, et en multipliant la loi de friction utilisée par un facteur correctif. Ce facteur correctif permet d'un point de vue mathématique d'obtenir un modèle consistant à l'ordre 1 avec la rhéologie. D'un point de vue physique, il induit une légère atténuation de l'augmentation du coefficient de friction basale avec la vitesse moyenne. La consistance du modèle à l'ordre 1 avec la rhéologie compressible considérée permet de calculer le nombre de Froude critique pour l'instabilité des ondes de surfaces de la rhéologie à partir du modèle linéarisé. Une version du modèle avec diffusion a également été proposée, en suivant la même méthode qu'au chapitre 3. Une formule permettant de reconstruire la hauteur et la fraction volumique solide

moyenne de manière consistante à partir de la solution a été donnée. Ce modèle possède les mêmes limitations que le modèle proposé au chapitre 3. En particulier, lorsque l'écoulement est fortement instationnaire et/ou non uniforme, ou lorsque la pente n'est pas comprise entre les paramètres μ_1 et μ_2 , la validité du modèle est discutable. Comme cela a été fait au chapitre 4 pour le modèle basé sur la rhéologie incompressible, une version régularisée de ce modèle pourrait être écrite afin de proposer un système d'équations qui reste cohérent et prédictif dans une large gamme de régimes. Cela permettrait notamment d'avoir un modèle valide à la fois sur des petites pentes, pour lesquelles la transition vers l'arrêt s'accompagne d'une compaction du milieu, et sur les grandes pentes, pour lesquelles l'écoulement est fortement accéléré, et une dilatation importante du milieu peut avoir lieu. Dans ce régime accéléré, la vitesse de glissement au fond peut se révéler importante, et utiliser une condition de non glissement paraît peu compatible avec les observations expérimentales. Une version encore plus élaborée du modèle pourrait être écrite en prenant en compte une vitesse de glissement au fond. Cette version pourrait être utilisée pour modéliser les ressauts granulaires qui se forment suite à l'impact de l'écoulement sur un obstacle. Des comparaisons quantitatives pourraient être réalisées afin de confirmer la validité du modèle.

Dans la troisième partie de la thèse, on a étudié un modèle jouet de suspension granulaire. Ce modèle est analogue aux équations de Navier-Stokes sans pression, avec un coefficient de diffusion non-linéaire (dépendant de la fraction volumique solide), qui diverge lorsque la fraction volumique solide s'approche d'une valeur maximale. Ce modèle présente donc quelques propriétés des suspensions granulaires qui ont été établies expérimentalement, et qui sont liées au phénomène de congestion (aussi appelé jamming). On a tout d'abord montré l'existence de solutions particulières de type ondes progressives, qui connectent un état congestionné avec un état non-congestionné. En particulier, la fraction volumique solide de l'onde progressive tend en moins l'infini vers la fraction volumique maximale. On a ensuite démontré la stabilité asymptotique de ces solutions. La preuve s'appuie sur plusieurs idées clefs. On a commencé par reformuler le système dans les coordonnées lagrangiennes de masse, ce qui permet de simplifier les calculs en faisant disparaître certains termes non-linéaires liés au transport. De plus, on a considéré des variables intégrées, pour lesquelles la perturbation satisfait une estimation d'énergie cruciale pour la stabilité aux temps longs. On a ensuite identifié un espace d'énergie adapté, faisant intervenir des poids dépendant de l'onde progressive considérée et dans lequel la fraction volumique solide des perturbations n'excède pas la fraction maximale. Ce travail pourrait être étendu à des modèles de suspensions plus élaborés, prenant notamment en compte la pression au sein du milieu, et éventuellement d'autres caractéristiques, comme un écoulement multi-dimensionnel.

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